EDN: UVEYUR
УДК 517.10

Collatz Hypothesis and Planck's Black Body Radiation

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Received 10.02.2023, received in revised form 13.01.2024, accepted 01.03.2024

Abstract. The Collatz conjecture is considered and the density of values is compared to Planck's black body radiation, showing a remarkable agreement with each other. We also briefly discuss a generalisation of Collatz conjecture.

Keywords: Collatz conjecture, Black body radiation, Collatz conjecture generalisation.

Citation: N. Fabiano, N. Mirkov, St. Radenović, Collatz Hypothesis and Planck's Black Body Radiation, J. Sib. Fed. Univ. Math. Phys., 2024, 17(3), 408–414. EDN: UVEYUR.

1. The Collatz conjecture

The Collatz conjecture starts from a very simple function. For $N \in \mathbb{N}$ define $C(N)$ as

$$C(N) = \begin{cases} \frac{N}{2} & \text{if } N \text{ even,} \\ 3N+1 & \text{if } N \text{ odd.} \end{cases} \tag{1}$$

This function applied recursively creates a sequence. Translating $C(N)$ to a sequence $\{a_i\}_{i\in\mathbb{N}}$, applying recursively the operation starting from a positive integer N one could write a_i as follows:

$$a_i = \begin{cases} N & \text{for } i = 0 \\ C(a_{i-1}) & \text{for } i > 0 \ , \end{cases} \tag{2}$$

so that $a_i = [C(N)]^i$.

For instance, starting from $N = 7$, the obtained sequence is

$$7, 22, 11, 34, 17, 52, 26, 13, 40, 20, 10, 5, 16, 8, 4, 2, 1 \ ,$$

while starting from $N = 1236$ the sequence is

$$1236, 618, 309, 928, 464, 232, 116, 58, 29, 88, 44, 22, 11, 34, 17, 52, 26, 13, 40,$$
$$20, 10, 5, 16, 8, 4, 2, 1 \ .$$

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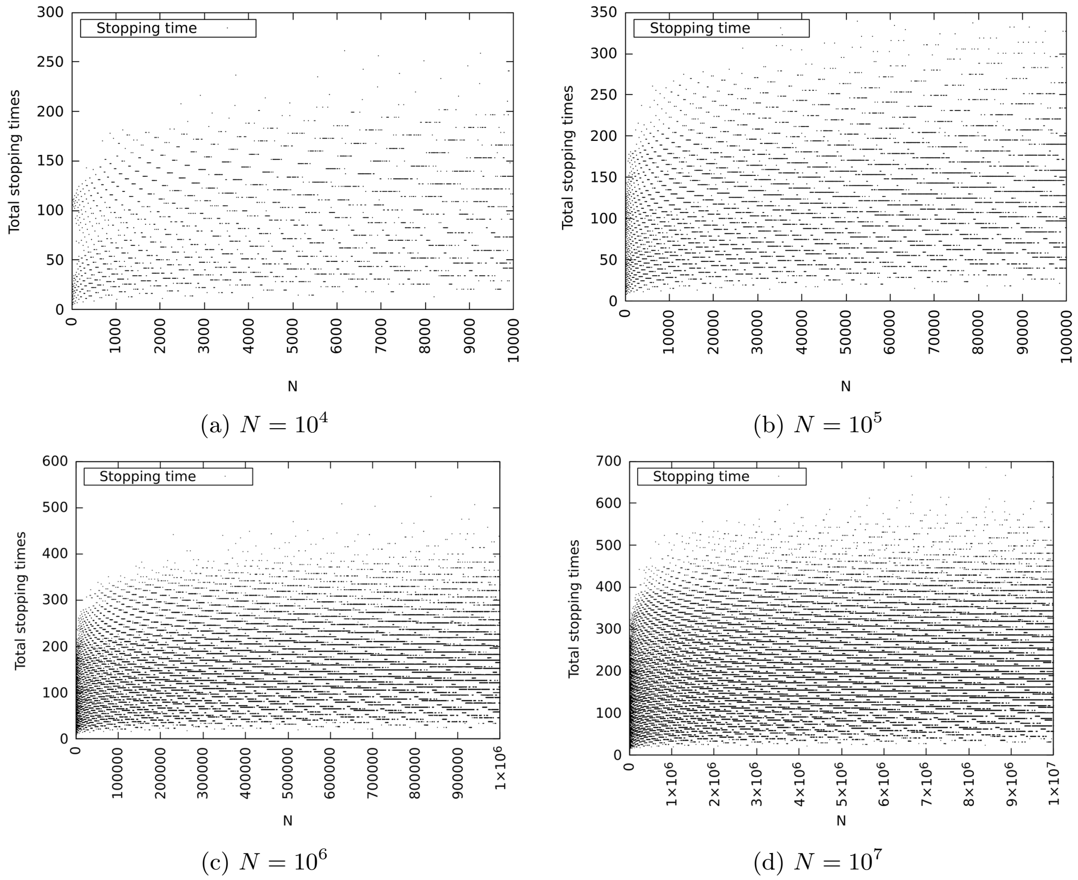
The sequence is concluded when it reaches the number 1. For the examples above, for 7 the sequence is concluded after 17 steps, while 27 steps are needed for 1236. Clearly, for a number of the type $N = 2^k$, $k \in \mathbb{N}$, 1 is reached after k steps. The number of steps necessary for reaching 1 is called total stopping time.

The conjecture of Collatz (1937) [1] states that function (1) starting from any natural number has a finite stopping time. Until today the conjecture has not been neither proved nor disproved.

There is plenty of literature on the subject, see for instance [2–6].

We are here concerned with the problem of total stopping times and its distribution. In Fig. 1 we have shown total stopping times for different starting values of N.



(a) $N = 10^4$ (b) $N = 10^5$

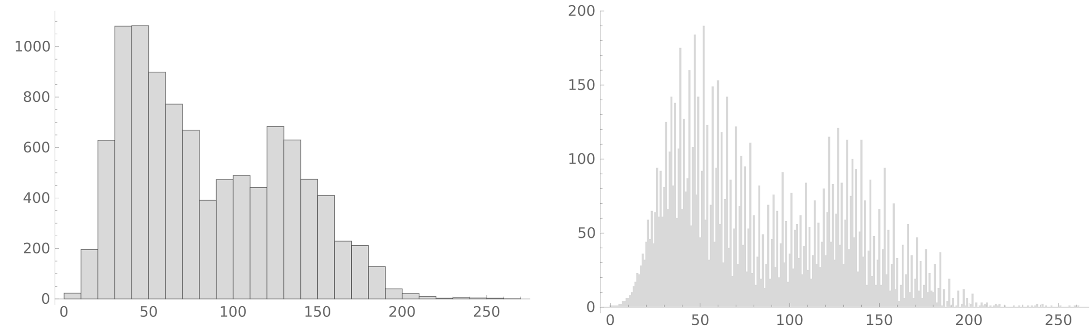
(c) $N = 10^6$ (d) $N = 10^7$

Fig. 1. Total stopping times for different starting N

A different kind of plot for total stopping times is shown in Fig. 2. For fixed N, total stopping times are shown with respect to frequencies. The latter results suggest a parallel to a well-known problem in physics.

2. Planck's radiation

Suppose to have electromagnetic radiation (that is, photons) at equilibrium inside a cavity of volume V at a temperature T. This system is known as "blackbody cavity". As is well-known, the free electromagnetic field can be written as a sum of harmonic oscillators at a fixed frequency ν. From the quantum theory each oscillator, that is each photon of frequency ν can only have



(a) Bin width=10 (b) Bin width=1

Fig. 2. Histograms for $N = 10^4$ of total stopping times with respect to frequencies for different bin widths

energies $\left(n' + \frac{1}{2}\right)h\nu$, for $n' \in \mathbb{N}$. The partition function of this system at the temperature T, $\beta = 1/k_B T$ can be written as

$$Z = \sum_{n'=0}^{+\infty} \exp(-\beta h\nu n') = \frac{1}{1 - \exp(-\beta h\nu)} \ . \tag{3}$$

The average energy for a photon of frequency ν is given by

$$\langle E \rangle = -\frac{d}{d\beta} \ln Z = \frac{h\nu \exp(-\beta h\nu)}{1 - \exp(-\beta h\nu)} = \frac{h\nu}{\exp(\beta h\nu) - 1} = \frac{h\nu}{\exp[h\nu/(k_B T)] - 1} \ . \tag{4}$$

The photon of frequency ν has a momentum $\vec{p} = (h/2\pi)\vec{k}$, and $k = |\vec{k}| = (2\pi/c)\nu$. For a volume V the number of momenta between k and $k + dk$ is given by

$$\frac{V}{(2\pi)^3} 8\pi k^2 dk = 8\pi \frac{V}{c^3} \nu^2 d\nu \ .$$

The internal energy U of the system is given by

$$U = 8\pi \frac{V}{c^3} \int_0^{+\infty} \langle E \rangle \nu^2 d\nu = 8\pi \frac{V}{c^3} \int_0^{+\infty} \frac{h\nu^3}{\exp[h\nu/(k_B T)] - 1} d\nu \ ,$$

so that the internal energy per unit volume is

$$\frac{U}{V} = \int_0^{+\infty} u(\nu, T) d\nu \ , \tag{5}$$

with

$$u(\nu, T) = 8\pi \left(\frac{h}{c^3}\right) \frac{\nu^3}{\exp[h\nu/(k_B T)] - 1} \ , \tag{6}$$

this is famous Planck's radiation law of photon energy density at frequency ν [7]. Performing integral (5) one observes that U/V increases with temperature as T^4, obtaining the Stefan–Boltzmann law for the power radiated from a black body.

Planck's function (6) has, remarkably, the same behaviour as the total stopping times with respect to frequencies shown in Fig. (2).

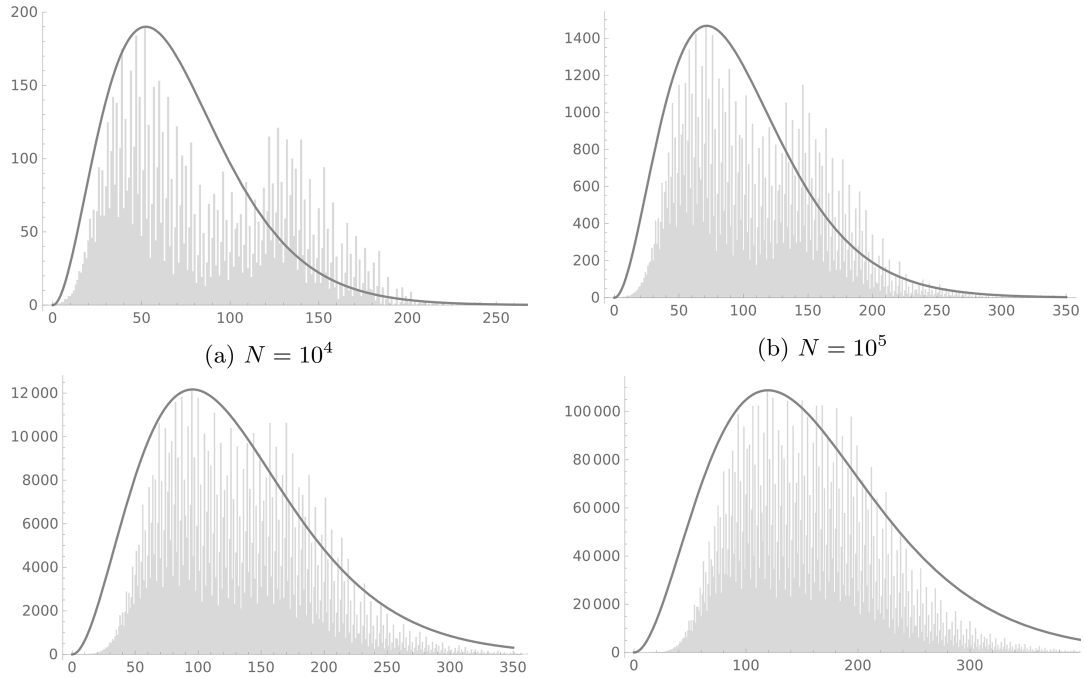
In Tab. (1) we have shown the equivalence between Collatz distribution of total stopping times and Planck's radiation density.

Table 1. Translation table Collatz–Planck

Collatz	$\longleftrightarrow$	Planck
Frequency	$\longleftrightarrow$	Photon frequency
Total stopping time	$\longleftrightarrow$	Black body radiation density

The results of the comparison of total stopping times and Planck's radiation density are shown in Fig. (3) for different values of starting N.

One could observe that the agreement between the two functions increases with increasing N, and also that Planck's radiation overestimates a little the decrease of stopping times with respect to N.



(a) $N = 10^4$ (b) $N = 10^5$

(c) $N = 10^6$ (d) $N = 10^7$

Fig. 3. Histograms: Total stopping times with respect to frequencies; Functions: Planck's black body radiation density with respect to photon frequency

From those results, formula (6), and with the aid of Tab. 1 we could infer that total stopping time goes to zero with increasing photon frequency ν at least as

$$\text{Total stopping time} \sim \nu^3 \exp(-\nu) \ . \tag{7}$$

Fig. (4) show the scaling of total stopping times with respect to the logarithm of starting N, $n = \log(N)(\approx 2.3 \ln(N))$, and the scaling of temperature T of Planck's radiation density with respect to n, respectively. The temperature is related to the longest stopping time, that is the peak of graphs in Fig. (3). Fig. (4a) shows a fairly linear scaling of total stopping times with the logarithm of starting N, that is

$$\text{Total stopping times} \sim \log(N) \ . \tag{8}$$

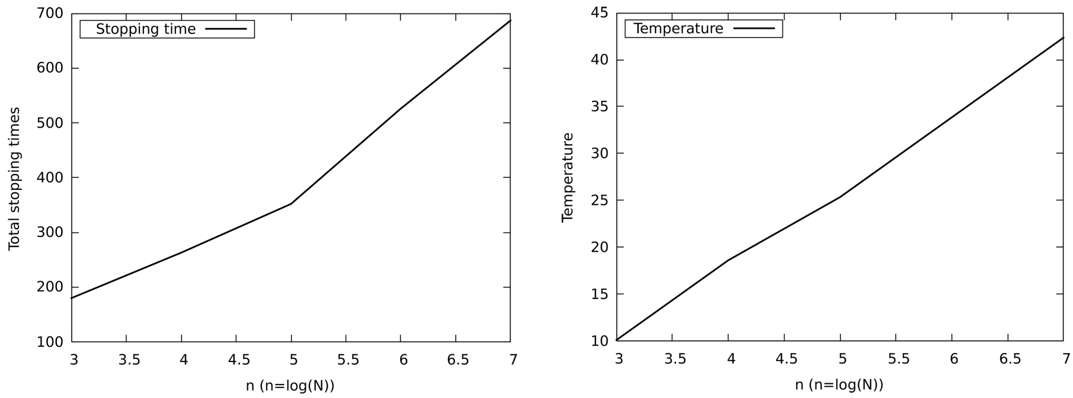
Fig. (4b) shows an almost perfect linear scaling for temperatures, or maximum values of total stopping times with the logarithm of starting N:

$$\text{Temperatures} \sim \log(N)\ . \tag{9}$$

From Stefan–Boltzmann's law we could also deduce that the sum of total stopping times scales with the fourth power of $\log(N)$:

$$\sum (\text{Total stopping times}) \sim (\log(N))^4\ . \tag{10}$$

All these results, although they are not proofs, show that for a finite N total stoppings times are finite, and their sum is finite as well, thus supporting Collatz's conjecture.



(a) Total stopping times (b) Temperatures

Fig. 4. (a): Scaling of total stopping times with respect to starting value of n, $n = \log(N)$; (b): Scaling of temperatures with respect to starting value of n, $n = \log(N)$.

3. Collatz generalisation

Define the generalisation of Collatz $C(q, N)$ as

$$C(q,N) = \begin{cases} \dfrac{N}{2} & \text{if } N \text{ even,} \\ qN+1 & \text{if } N \text{ odd.} \end{cases} \tag{11}$$

wich has fixed points $x = 0$ and $x = -1/(q-1)$ respectively. For even N behaves as usual, so we have to focus only on odd N. Let q be even, $q = 2j$. For odd $N = 2m + 1$ one has

$$2j(2m+1)+1 = 2[j(2m+1)]+1$$

which is again odd and grows without bounds. Therefore, q has to be odd, $q = 2j + 1$.
For $N = 2m + 1$ we obtain

$$(2j+1)(2m+1)+1 = 2[(2j+1)m+(j+1)] \tag{12}$$

which is even. Using the notation $\to$ as "transforms to" we obtain Tab. 2. Actually the original Collatz function loops on the sequence $4, 2, 1$ for any N, so one could say that the conjecture is verified when any $N \in \mathbb{N}$ enters this loop.

For $q > 3$ the situation is different, probably due to the fact that observing Tab. (2), and (12), the ratio

$$\frac{2j+1}{j+1}$$

is larger than 3/2 so the sequence reaches fewer odd numbers with respect to even numbers.

Table 2. Transformation table for Collatz generalisation

j	q	$(2m+1) \to$
1	3	$2(3m+2)$
2	5	$2(5m+3)$
3	7	$2(7m+4)$
4	9	$2(9m+5)$

In the case $q = 5$ we have various different loops than the usual $4, 2, 1$.

For $N = 5$:

$$5, 26, 13, 66, 33, 166, 83, 416, 208, 104, 52, 26, 13, \ldots$$

For $N = 13$:

$$13, 66, 33, 166, 83, 416, 208, 104, 52, 26, 13, 66, \ldots$$

For $N = 15$:

$$15, 76, 38, 19, 96, 48, 24, 12, 6, 3, 16, 8, 4, 2, 1, 6, 3, 16, 8, 4, \ldots$$

For $N = 17$:

$$17, 86, 43, 216, 108, 54, 27, 136, 68, 34, 17, 86, \ldots$$

While for other values $C(q, N)$ diverges, like $N = 7, 9, 11, 14, 18, \ldots$.

For $q > 5$ the situation worsens drastically, and $C(q, N)$ diverges already for $N = 5$, $C(q, 5)$, for values of $q = 9, 11, 13, 15, 17, 19, 21, \ldots, 211, \ldots$.

One could conjecture that only the Collatz case $q = 3$ has no divergencies and an unique loop, $4, 2, 1$, while for $q > 3$ there exist more loops separated by some divergent points.

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Гипотеза Коллатца и излучение черного тела Планка

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Аннотация. Рассматривается гипотеза Коллатца, и плотность значений сравнивается с излучением черного тела Планка, демонстрируя замечательное согласие друг с другом. Мы также кратко обсудим обобщение гипотезы Коллатца.

Ключевые слова: гипотеза Коллатца, излучение черного тела, обобщение гипотезы Коллатца.