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On Properties of the Second Type Matrix Ball $B^{(2)}_{m,n}$ from Space $\mathbb{C}^n[m \times m]$

Uktam S. Rakhmonov*
Tashkent State Technical University
Tashkent, Uzbekistan
Jonibek Sh. Abdullayev†
National University of Uzbekistan
Tashkent, Uzbekistan

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Abstract. The automorphisms of the matrix ball associated with the classical domains of the second type are described in this paper. The properties of the second type matrix ball $B^{(2)}_{m,n}$ are studied.

Keywords: classical domain, matrix ball, automorphism of the matrix ball, volume, Shilov's boundary.

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1. Introduction, preliminaries and problem statement

The theory of functions of several complex variables, or multidimensional complex analysis, currently is rather rigorously developed (see [1–4]). At the same time, many questions of classical complex analysis still do not have unambiguous multidimensional analogues. The matrix approach to the presentation of the theory of multidimensional complex analysis was widely used (see [5–8]).

In 1935 E.Cartan proved that there are only six possible types of classical domains, including irreducible, homogeneous, bounded, symmetric domains, four of them K_1, K_2, K_3 and K_4 have the form

$$K_1 = \left\{ Z \in \mathbb{C}\,[m \times k] : I^{(m)} - ZZ^* > 0 \right\},$$

$$K_2 = \left\{ Z \in \mathbb{C}\,[m \times m] : I^{(m)} - Z\overline{Z} > 0, \quad \forall Z' = Z \right\},$$

$$K_3 = \left\{ Z \in \mathbb{C}\,[m \times m] : I^{(m)} + Z\overline{Z} > 0, \quad \forall Z' = -Z \right\},$$

$$K_4 = \left\{ z \in \mathbb{C}^n : |zz'|^2 + 1 - 2\overline{z}z' > 0, \quad |zz'| < 1 \right\}.$$

Here $I^{(m)}$ is the identity matrix of order m, Z^* is the complex conjugate of transposed matrix Z' ($H > 0$ means that hermitian matrix H is positive definite).

The dimensions of these domains are equal to mk, $m(m+1)/2$, $m(m-1)/2$, n, respectively.

*uktam_rakhmonov@mail.ru https://orcid.org/0000-0001-8926-9505
†jonibek-abdullayev@mail.ru https://orcid.org/0000-0001-8950-2135

All these domains are biholomorphically non-equivalent, therefore, complex analysis is constructed differently for each of them.

It should be noted[‡] that domain K_4 is reducible for $n = 2$ (see [6]). In contrast, the other domains of all four types are irreducible, but the same domains can be found. Switching the places of m and k does not change domains in K_1. Further, the unit circle of the complex plane is obtained when $m = k = 1$ in K_1, $m = 1$ in K_2, $m = 2$ in K_3 and $n = 1$ in K_4. When $m = 3$, $k = 1$ in domain K_1 then K_1 coincides with domain K_3 including $m = 3$. When $m = k = 2$ in domain K_1 then K_1 coincides with domain K_4 including $n = 4$. When $m = 2$ in domain K_2 then K_2 coincides with domain K_4 including $n = 3$. Thus, we obtain different irreducible domains if we demand $m \geqslant k$ in K_1, $m \geqslant 2$ in K_2, $m \geqslant 4$ in K_3 and $n \geqslant 5$ in K_4. So, the number $\psi(n)$ of classes of irreducible bounded symmetric domains of an n-dimensional complex space is equal to the total number of representations of n in one of the following forms

$$K_1 : n = mk \ \ (m \geqslant k),$$

$$K_2 : n = \frac{1}{2}m(m+1) \ \ (m \geqslant 2),$$

$$K_3 : n = \frac{1}{2}m(m-1) \ \ (m \geqslant 4),$$

$$K_4 : n = m \quad (n \geqslant 5),$$

$$K_5, K_6 : n = 16, \ n = 27.$$

All irreducible domains obtained in this way are topologically (but not analytically) equivalent to the n-dimensional complex space.

Let us consider the space of m^2 complex variables denoted by $\mathbb{C}^{m^2}$. Points Z of this space can be represented conveniently as a square $[m \times m]$ matrices, i.e., in the form $Z = (z_{ij})_{i,j=1}^m$. With this representation of points the space $\mathbb{C}^{m^2}$ is denoted by $\mathbb{C}[m \times m]$. The direct product $\underbrace{\mathbb{C}[m \times m] \times \cdots \times \mathbb{C}[m \times m]}_{n}$ of n copies of $[m \times m]$ matrix spaces is denoted by $\mathbb{C}^n[m \times m]$.

Let $Z = (Z_1, Z_2, \ldots, Z_n)$ be a vector composed of square matrices Z_j of order m considered over the field of complex numbers $\mathbb{C}$. We can assume that Z is an element of the set $\mathbb{C}^n[m \times m] \cong \mathbb{C}^{nm^2}$.

The matrix «scalar product» is defined as $(Z, W \in \mathbb{C}^n[m \times m])$

$$\langle Z, W \rangle = Z_1 W_1^* + Z_2 W_2^* + \cdots + Z_n W_n^*.$$

It is known that matrix balls $B_{m,n}^{(1)}$, $B_{m,n}^{(2)}$ and $B_{m,n}^{(3)}$ of the first, second, and third types have the following forms, respectively (see [9–11]):

$$B_{m,n}^{(1)} = \{(Z_1, \ldots, Z_n) = Z \in \mathbb{C}^n\,[m \times m] : I - \langle Z, Z \rangle > 0\},$$

$$B_{m,n}^{(2)} = \{Z \in \mathbb{C}^n\,[m \times m] : I - \langle Z, Z \rangle > 0 \ \ \forall\, Z_\nu' = Z_\nu, \ \nu = 1, \ldots, n\}$$

and

$$B_{m,n}^{(3)} = \left\{(Z \in \mathbb{C}^n\,[m \times m] : I + \langle Z, Z \rangle > 0 \ \ \forall\, Z_\nu^{'} = -Z_\nu, \ \nu = 1, \ldots, n\right\}.$$

[‡]When $n=2$, a homogeneous bounded domain is equivalent to the domain $K = \left\{\zeta \in \mathbb{C}^2 : \max\left(|\zeta_1| < 1, |\zeta_2| < 1\right)\right\}$, after the change of variables: $z_1 = \frac{\zeta_1 + \zeta_2}{2}, z_2 = \frac{i(\zeta_1 - \zeta_2)}{2}$.

The skeletons (Shilov boundaries) of the matrix balls $B_{m,n}^{(k)}$ are denoted by $X_{m,n}^{(k)}$, $k = 1, 2, 3$, i.e.,

$$X_{m,n}^{(1)} = \{Z \in \mathbb{C}^n [m \times m] : \langle Z, Z \rangle = I\},$$

$$X_{m,n}^{(2)} = \{Z \in \mathbb{C}^n [m \times m] : \langle Z, Z \rangle = I,\ Z'_v = Z_\nu,\ \nu = 1, 2, \ldots, n\},$$

$$X_{m,n}^{(3)} = \left\{Z \in \mathbb{C}^n [m \times m] : I + \langle Z, Z \rangle = 0,\ Z_\nu^{'} = -Z_\nu,\ \nu = 1, 2, \ldots, n\right\}.$$

Note that $B_{1,1}^{(1)}$, $B_{1,1}^{(2)}$ and $B_{2,1}^{(3)}$ are unit disks, and $X_{1,1}^{(1)}$, $X_{1,1}^{(2)}$, $X_{2,1}^{(3)}$ are unit circles in the complex plane $\mathbb{C}$.

If $n = 1$, $m > 1$ then domains $B_{m,1}^{(k)}$, $k = 1, 2, 3$ are the classical domains of the first, second and the third type (according to the classification of E. Cartan (see [5])). The skeletons $X_{m,1}^{(1)}$, $X_{m,1}^{(2)}$, and $X_{m,1}^{(3)}$ are unitary, symmetric unitary and skew-symmetric unitary matrices, respectively.

The first type of matrix ball was considered by A. G. Sergeev (see [11,26]), G. Khudayberganov (see [12,13]) and S. Kosbergenov (see [14,15]). The volume of a matrix ball of the first type and its skeleton is studied in [16]. Holomorphic automorphisms for a matrix ball of the first type are described in [17]. The integral formulas for the matrix ball of the second type were studied by G. Khudayberganov and Z. Matyakubov [18,19] and the third type of the matrix ball was studied by G. Khudayberganov, U. Rakhmonov, and the integral formulas were found [20,21]. We recall that a bounded domain $D \subset \mathbb{C}^n$ is called classical if the complete group of its holomorphic automorphisms is a classical Lie group and transitive on it. The biholomorphic equivalence of bounded domains in $\mathbb{C}^n$ to their indicatrices for the Carathéodory and Kobayashi metrics was studied [32]. From this, in particular, a description of that domains can be obtained when indicatrices are classical domains. It was proved that first, second and third type matrix balls in space $\mathbb{C}^n[m \times m]$ are equivalent biholomorphically to Siegel domains of the second type [27–29]. However, the question of whether matrix balls $B_{m,n}^{(1)}$, $B_{m,n}^{(2)}$ and $B_{m,n}^{(3)}$ are the classical domains still remains open.

The problem of the holomorphic extendability of a function to a matrix ball, given on a piece of its skeleton was discussed [26]. For this purpose complete orthonormal systems in the matrix ball were used. The total volumes of a matrix ball of the third type and a generalized Lie ball were calculated [22]. The full volumes of these domains are necessary for finding the kernels of the integral formulas for these domains (the Bergman, Cauchy–Szegő kernels, Poisson kernels, etc. [14, 19, 23, 30]). In addition, they are used for the integral representation of a holomorphic function on these domains, in the mean value theorem and in other important concepts. Volumes of classical supermanifolds such as supersphere, complex projective superspace, and the Stifel and Grassmann supermanifolds were calculated with respect to natural metrics of symplectic structures. It was shown that formulas for volumes of these supermanifolds can be obtained by analytic continuation of the parameters from the formulas for the volumes of the corresponding ordinary varieties (see [24]).

In this paper we describe automorphisms of the matrix ball associated with classical domains of the second type, and also study the properties of the second type matrix ball. An automorphism of the second type matrix ball and the characteristic shape of this ball were studied [10]. Writing automorphism in this form causes inconvenience in applying it to practical issues. Therefore, we consider automorphisms of a matrix ball of the second type which are convenient for calculations. In addition, the total volume of the skeleton of this ball is calculated.

2. Automorphisms for a matrix ball of the second type

Let $B^{(2)}_{m,n}$ be a matrix ball of the second type and $X^{(2)}_{m,n}$ is its skeleton. The following lemma describes some properties of a matrix ball of the second type [18].

Lemma 1. *A matrix ball $B^{(2)}_{m,n}$ has the following properties:*

1) $B^{(2)}_{m,n}$ is a bounded domain;

2) $B^{(2)}_{m,n}$ is a full circular domain;

3) $B^{(2)}_{m,n}$ and its skeleton $X^{(2)}_{m,n}$ are invariants under unitary transformations.

It is known that automorphism $B^{(2)}_{m,1}$ which maps the point $P \in B^{(2)}_{m,1}$ to the point 0 has the form [8]

$$W = R(Z - P)(I - \bar{P}Z)^{-1}\bar{R}^{-1},$$

where R is $[m \times m]$ matrix

$$\bar{R}(I - \bar{P}P')R' = I.$$

Our goal is to find automorphisms for a matrix ball of the second type. Let us consider the desired automorphism in the form

$$W_k = \left(A_{00} + \sum_{j=1}^{n} Z_j A_{j0}\right)^{-1}\left(A_{0k} + \sum_{j=1}^{n} Z_j A_{jk}\right), \ \ k = 1, \dots, n. \tag{1}$$

We need to find the coefficients A_{ij} so that map (1) is an automorphism of the matrix ball of the second type.

Let us introduce the following notation of block square matrices of order $n + 1$

$$A = \begin{pmatrix} A_{00} & A_{01} & \dots & A_{0n} \\ A_{10} & A_{11} & \dots & A_{1n} \\ \dots & \dots & \dots & \dots \\ A_{n0} & A_{n1} & \dots & A_{nn} \end{pmatrix}, \ \ H = \begin{pmatrix} I^{(m)} & 0 & \dots & 0 \\ 0 & -I^{(m)} & \dots & 0 \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & -I^{(m)} \end{pmatrix},$$

where A_{ij} are square matrices of order m.

The following statement holds.

Theorem 1. *Mapping (1) is an automorphism of the matrix ball $B^{(2)}_{m,n}$ if and only if coefficients A_{ij}, $i, j = 0, 1, 2, \dots, n$ satisfy the following relations:*

$$AHA^* = H, \ \ A_{sk}A'_{j0} = A_{j0}A'_{jk}, \ \ s = 0, \dots, n; \ j, k = 0, \dots, n. \tag{2}$$

Proof. This theorem is proved in several stages, according to the properties of a matrix ball of the second type.

1^0. Let us consider a linear transformation

$$\omega_0 = \sum_{j=0}^{n} \zeta_j A_{j0}, \ \ \omega_k = \sum_{j=0}^{n} \zeta_j A_{jk}, \ \ k = 1, \dots, n, \tag{3}$$

where matrix A satisfies relations (2). Then we have

$$AHA^* = \begin{pmatrix} A_{00} & A_{01} & \dots & A_{0n} \\ A_{10} & A_{11} & \dots & A_{1n} \\ \dots & \dots & \dots & \dots \\ A_{n0} & A_{n1} & \dots & A_{nn} \end{pmatrix} \begin{pmatrix} I^{(m)} & 0 & \dots & 0 \\ 0 & -I^{(m)} & \dots & 0 \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & -I^{(m)} \end{pmatrix} \times$$

$$\times\begin{pmatrix} A^*_{00} & A^*_{10} & \dots & A^*_{n0} \\ A^*_{01} & A^*_{11} & \dots & A^*_{n1} \\ \dots & \dots & \dots & \dots \\ A^*_{0n} & A^*_{1n} & \dots & A^*_{nn} \end{pmatrix} = \begin{pmatrix} I^{(m)} & 0 & \dots & 0 \\ 0 & -I^{(m)} & \dots & 0 \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & -I^{(m)} \end{pmatrix},$$

$$\begin{pmatrix} A_{00} & -A_{01} & \dots & -A_{0n} \\ A_{10} & -A_{11} & \dots & -A_{1n} \\ \dots & \dots & \dots & \dots \\ A_{n0} & -A_{n1} & \dots & -A_{nn} \end{pmatrix}\begin{pmatrix} A^*_{00} & A^*_{10} & \dots & A^*_{n0} \\ A^*_{01} & A^*_{11} & \dots & A^*_{n1} \\ \dots & \dots & \dots & \dots \\ A^*_{0n} & A^*_{1n} & \dots & A^*_{nn} \end{pmatrix} = \begin{pmatrix} I^{(m)} & 0 & \dots & 0 \\ 0 & -I^{(m)} & \dots & 0 \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & -I^{(m)} \end{pmatrix}$$

$$\begin{pmatrix} A_{00}A^*_{00}-\dots-A_{0n}A^*_{0n} & A_{00}A^*_{10}-\dots-A_{0n}A^*_{1n} & \dots & A_{00}A^*_{n0}-\dots-A_{0n}A^*_{nn} \\ A_{10}A^*_{00}-\dots-A_{1n}A^*_{0n} & A_{10}A^*_{10}-\dots-A_{1n}A^*_{1n} & \dots & A_{10}A^*_{n0}-\dots-A_{1n}A^*_{nn} \\ \dots & \dots & \dots & \dots \\ A_{n0}A^*_{00}-\dots-A_{nn}A^*_{0n} & A_{n0}A^*_{10}-\dots-A_{nn}A^*_{1n} & \dots & A_{n0}A^*_{n0}-\dots-A_{nn}A^*_{nn} \end{pmatrix} =$$

$$= \begin{pmatrix} I^{(m)} & 0 & \dots & 0 \\ 0 & -I^{(m)} & \dots & 0 \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & -I^{(m)} \end{pmatrix} \Rightarrow$$

$$\Rightarrow \begin{cases} A_{00}A^*_{00} - \sum\limits_{s=1}^{n} A_{0s}A^*_{0s} = I^{(m)}, \\ A_{j0}A^*_{k0} = \sum\limits_{s=1}^{n} A_{js}A^*_{ks}, \ \ j \neq k, \\ A_{j0}A^*_{j0} - \sum\limits_{s=1}^{n} A_{js}A^*_{js} = -I^{(m)}, \ \ j \geqslant 1. \end{cases} \tag{4}$$

2^0. Let matrix row $\zeta = (\zeta_0, \zeta_1, \dots, \zeta_n)$ covers all matrices consisting of m rows and $(n+1)m$ columns such that $\zeta H \zeta^* > 0$. Then

$$\zeta H \zeta^* = \begin{pmatrix} \zeta_0 & \zeta_1 & \dots & \zeta_n \end{pmatrix}\begin{pmatrix} I^{(m)} & 0 & \dots & 0 \\ 0 & -I^{(m)} & \dots & 0 \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & -I^{(m)} \end{pmatrix}\begin{pmatrix} \zeta^*_0 \\ \zeta^*_1 \\ \dots \\ \zeta^*_n \end{pmatrix} =$$

$$= \begin{pmatrix} \zeta_0 & -\zeta_1 & \dots & -\zeta_n \end{pmatrix}\begin{pmatrix} \zeta^*_0 \\ \zeta^*_1 \\ \dots \\ \zeta^*_n \end{pmatrix} = \zeta_0\zeta^*_0 - \zeta_1\zeta^*_1 - \dots - \zeta_n\zeta^*_n > 0 \Rightarrow$$

$$\Rightarrow \zeta_0\zeta^*_0 > \zeta_1\zeta^*_1 + \dots + \zeta_n\zeta^*_n \geqslant 0.$$

Providing $\zeta H \zeta^* > 0$, matrix ζ_0 is not degenerate since otherwise there would be a non-zero m-dimensional vector x such that $x\zeta_0 = 0$.

We have a contradiction since

$$0 = x\zeta_0\zeta^*_0x^* > x(\zeta_1\zeta^*_1 + \dots + \zeta_n\zeta^*_n)x^* \geqslant 0.$$

3^0. Now we consider the following matrices

$$Z_k = \zeta_0^{-1}\zeta_k, \ \ k = 1, \dots, n.$$

We obtain the following inequality from condition $\zeta H\zeta^* > 0$

$$\zeta H\zeta^* = \begin{pmatrix} \zeta_0 & \zeta_1 & \dots & \zeta_n \end{pmatrix} \begin{pmatrix} I^{(m)} & 0 & \dots & 0 \\ 0 & -I^{(m)} & \dots & 0 \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & -I^{(m)} \end{pmatrix} \begin{pmatrix} \zeta_0^* \\ \zeta_1^* \\ \dots \\ \zeta_n^* \end{pmatrix} =$$

$$= \begin{pmatrix} \zeta_0 & -\zeta_1 & \dots & -\zeta_n \end{pmatrix} \begin{pmatrix} \zeta_0^* \\ \zeta_1^* \\ \dots \\ \zeta_n^* \end{pmatrix} = \zeta_0\zeta_0^* - \zeta_1\zeta_1^* - \dots - \zeta_n\zeta_n^* =$$

$$= \zeta_0\big(I - \zeta_0^{-1}\zeta_1\zeta_1^*(\zeta_0^*)^{-1} - \dots - \zeta_0^{-1}\zeta_n\zeta_n^*(\zeta_0^*)^{-1}\big)\zeta_0^* =$$

$$= \zeta_0\left(I^{(m)} - Z_1Z_1^* - \dots - Z_nZ_n^*\right)\zeta_0^* = \zeta_0\big(I - \langle Z, Z\rangle\big)\zeta_0^* > 0$$

$$\Rightarrow I^{(m)} - \langle Z, Z\rangle > 0, \text{ i.e., } Z \in B_{m,n}^{(2)}.$$

4^0. Using (3) we consider the vector

$$\omega = (\omega_0, \omega_1, \dots, \omega_n) = \zeta A$$

and multiply the block matrix by the right of the above-mentioned formula

$$\widetilde{A} = \begin{pmatrix} A_{00}^* & -A_{10}^* & \dots & -A_{n0}^* \\ -A_{01}^* & A_{11}^* & \dots & A_{n1}^* \\ \dots & \dots & \dots & \dots \\ -A_{0n}^* & A_{1n}^* & \dots & A_{nn}^* \end{pmatrix}.$$

Note that the product of block matrices is carried out according to the usual rules for the product of matrices. Since (4) is equivalent to the condition $A\widetilde{A} = I^{(m(n+1))}$, then we have

$$\omega\widetilde{A} = \zeta,$$

i.e., map (3) is invertible (under condition (2)) and the matrix defines the inverse map.

Hence,

$$\omega H\omega^* = \zeta AHA^*\zeta^* = \zeta H\zeta^* > 0. \tag{5}$$

5^0. Now we prove that map W_k is an automorphism. Obviously,

$$\omega H\omega^* = \begin{pmatrix} \omega_0 & \omega_1 & \dots & \omega_n \end{pmatrix} \begin{pmatrix} I^{(m)} & 0 & \dots & 0 \\ 0 & -I^{(m)} & \dots & 0 \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & -I^{(m)} \end{pmatrix} \begin{pmatrix} \omega_0^* \\ \omega_1^* \\ \dots \\ \omega_n^* \end{pmatrix} =$$

$$= \begin{pmatrix} \omega_0 & -\omega_1 & \dots & -\omega_n \end{pmatrix} \begin{pmatrix} \omega_0^* \\ \omega_1^* \\ \dots \\ \omega_n^* \end{pmatrix} = \omega_0\omega_0^* - \omega_1\omega_1^* - \dots - \omega_n\omega_n^* =$$

$$= \omega_0\big(I - \omega_0^{-1}\omega_1\omega_1^*(\omega_0^*)^{-1} - \dots - \omega_0^{-1}\omega_n\omega_n^*(\omega_0^*)^{-1}\big)\omega_0^* =$$

$$= \omega_0\left(I^{(m)} - W_1W_1^* - \dots - W_nW_n^*\right)\omega_0^* = \omega_0\big(I - \langle W, W\rangle\big)\omega_0^* > 0 \Rightarrow I - \langle W, W\rangle > 0.$$

Then transformation (3) generates a linear-fractional transformation

$$W_k = \omega_0^{-1}\omega_k = \left(\sum_{j=0}^{n}\zeta_j A_{j0}\right)^{-1}\left(\sum_{j=0}^{n}\zeta_j A_{jk}\right) = \left(\zeta_0 A_{00} + \sum_{j=1}^{n}\zeta_j A_{j0}\right)^{-1}\left(\zeta_0 A_{0k} + \sum_{j=1}^{n}\zeta_j A_{jk}\right) =$$
$$= \left(A_{00} + \sum_{j=1}^{n}\zeta_0^{-1}\zeta_j A_{j0}\right)^{-1}\zeta_0^{-1}\zeta_0\left(A_{0k} + \sum_{j=1}^{n}\zeta_0^{-1}\zeta_j A_{jk}\right) =$$
$$= \left(A_{00} + \sum_{j=1}^{n} Z_j A_{j0}\right)^{-1}\left(A_{0k} + \sum_{j=1}^{n} Z_j A_{jk}\right), \quad k = 1, \dots, n.$$

6^0. Let us show that matrices $W_k, k = 1, \dots, n$ are symmetric matrices. Let $W_k = \omega_0^{-1}\omega_k$ then $W_k' = \omega_k'(\omega_0')^{-1}$ and

$$W_k - W_k' = \omega_0^{-1}\omega_k - \omega_k'(\omega_0')^{-1} = \omega_0^{-1}(\omega_k\omega_0' - \omega_0\omega_k')(\omega_0')^{-1};$$

$$\omega_k\omega_0' - \omega_0\omega_k' = \sum_{j=0}^{n}\zeta_j A_{jk}\sum_{j=0}^{n}A_{j0}'\zeta_j' - \sum_{j=0}^{n}\zeta_j A_{j0}\sum_{j=0}^{n}A_{jk}'\zeta_j' =$$
$$= (\zeta_0 A_{0k} + \zeta_1 A_{1k} + \dots + \zeta_n A_{nk})(A_{00}'\zeta_0' + A_{10}\zeta_1' + \dots + A_{n0}'\zeta_n') -$$
$$-(\zeta_0 A_{00} + \zeta_1 A_{10} + \dots + \zeta_n A_{n0})(A_{0k}'\zeta_0' + A_{1k}\zeta_1' + \dots + A_{nk}'\zeta_n') =$$
$$= \zeta_0(A_{0k}A_{00}' - A_{00}A_{0k}')\zeta_0' + \zeta_0(A_{0k}A_{10}' - A_{10}A_{1k}')\zeta_1' + \dots +$$
$$+\zeta_0(A_{0k}A_{n0}' - A_{00}A_{0k}')\zeta_n' + \zeta_1(A_{1k}A_{00}' - A_{10}A_{0k}')\zeta_0' +$$
$$+\zeta_1(A_{1k}A_{10}' - A_{10}A_{1k}')\zeta_1' + \dots + \zeta_1(A_{1k}A_{n0}' - A_{10}A_{nk}')\zeta_n' + \dots +$$
$$+\zeta_n(A_{nk}A_{00}' - A_{n0}A_{0k}')\zeta_0' + \zeta_n(A_{nk}A_{10}' - A_{n0}A_{1k}')\zeta_1' + \dots +$$
$$+\zeta_n(A_{nk}A_{n0}' - A_{n0}A_{nk}')\zeta_n' = 0.$$

The last equality is valid by virtue of (2).

Theorem 1 is proved. □

Further, using relation $A\widetilde{A} = I^{(m(n+1))}$, we obtain $\widetilde{A}A = I^{(m(n+1))}$. It means that

$$\widetilde{A}A = \begin{pmatrix} A_{00}^* & -A_{10}^* & \dots & -A_{n0}^* \\ -A_{01}^* & A_{11}^* & \dots & A_{n1}^* \\ \dots & \dots & \dots & \dots \\ -A_{0n}^* & A_{1n}^* & \dots & A_{nn}^* \end{pmatrix}\begin{pmatrix} A_{00} & A_{01} & \dots & A_{0n} \\ A_{10} & A_{11} & \dots & A_{1n} \\ \dots & \dots & \dots & \dots \\ A_{n0} & A_{n1} & \dots & A_{nn} \end{pmatrix} = I^{(m(n+1))},$$

$$\begin{pmatrix} A_{00}^*A_{00} - \dots - A_{n0}^*A_{n0} & A_{00}^*A_{01} - \dots - A_{n0}^*A_{n1} & \dots & A_{00}^*A_{0n} - \dots - A_{n0}^*A_{nn} \\ -A_{01}^*A_{00} + \dots + A_{n1}^*A_{n0} & -A_{01}^*A_{01} + \dots + A_{n1}^*A_{n1} & \dots & -A_{01}^*A_{0n} + \dots + A_{n1}^*A_{nn} \\ \dots & \dots & \dots & \dots \\ -A_{0n}^*A_{00} + \dots + A_{nn}^*A_{n0} & -A_{0n}^*A_{01} + \dots + A_{nn}^*A_{n1} & \dots & -A_{0n}^*A_{0n} + \dots + A_{nn}^*A_{nn} \end{pmatrix} \Rightarrow$$

$$\Rightarrow \begin{cases} A_{00}^*A_{00} - \sum\limits_{j=1}^{n} A_{j0}^*A_{j0} = I^{(m)}, \\ A_{0k}^*A_{0j} = \sum\limits_{s=1}^{n} A_{sk}^*A_{sj}, \quad j \neq k, \\ A_{0k}^*A_{0k} - \sum\limits_{j=1}^{n} A_{jk}^*A_{jk} = -I^{(m)} \end{cases}. \qquad (6)$$

Now let the point $P = (P_1, \ldots, P_n) \in B^{(2)}_{m,n}$. Let us consider the mapping

$$W_k = R^{-1}(I^{(m)} - \langle Z, P\rangle)^{-1} \sum_{s=1}^{n} (Z_s - P_s) G_{sk}, \ k = 0, 1, \ldots, n \tag{7}$$

that transfers the point P to 0, where R, G_{sk} are arbitrary matrices.

Theorem 2. *For a mapping of form (7) to be an automorphism of a matrix ball of the second type it is necessary and sufficient that matrices R and G satisfy the following relations*

$$R^*(I^{(m)} - \langle P, P\rangle)R = I^{(m)}, \quad G^*(I^{(mn)} - P^*P)G = I^{(mn)}, \tag{8}$$

where G is a block matrix.

Proof. Necessity. Let mapping of form (7) be an automorphism of the matrix ball $B^{(2)}_{m,n}$ that maps the point P to 0. We have that

$$\begin{aligned} A_{00} = R, \quad A_{j0} = -P_j^* R, \quad j = 1, \ldots, n, \\ A_{jk} = G_{jk}, \quad j, k = 1, \ldots, n, \\ A_{0k} = -\sum_{s=1}^{n} P_s G_{jk}, \quad k = 1, \ldots, n, \end{aligned} \tag{9}$$

$$\begin{aligned} (1) \Rightarrow W_k &= \left(A_{00} + \sum_{j=1}^{n} Z_j A_{j0}\right)^{-1} \left(A_{0k} + \sum_{j=1}^{n} Z_j A_{jk}\right) = \\ &= \left(R - \sum_{j=1}^{n} Z_j P_j^* R\right)^{-1} \left(-\sum_{s=1}^{n} P_s G_{sk} + \sum_{j=1}^{n} Z_j G_{jk}\right) = \\ &= R^{-1}\left(I - \langle Z, P\rangle\right)^{-1} \sum_{s=1}^{n} \left(Z_s - P_s\right) G_{sk}. \end{aligned}$$

Taking into account (6) and (9), we obtain (8)

$$R^*R - \sum_{j=1}^{n} R^* P_j P_j^* R = I^{(m)} \ \Rightarrow \ R^*(I^{(m)} - \langle P, P\rangle)R = I^{(m)},$$

$$\sum_{s=1}^{n} G_{sk}^* P_s^* \sum_{s=1}^{n} P_s G_{sk} - \sum_{j=1}^{n} G_{jk}^* G_{jk} = -I^{(m)},$$

$$\sum_{s=1}^{n} G_{sk}^* P_s^* \sum_{s=1}^{n} P_s G_{sj} = \sum_{s=1}^{n} G_{sk}^* G_{sk}, j \neq k,$$

$$G^*(I^{(mn)} - P^*P)G = I^{(mn)},$$

$$G = \begin{pmatrix} G_{11} & G_{12} & \ldots & G_{1n} \\ G_{21} & G_{22} & \ldots & G_{2n} \\ \ldots & \ldots & \ldots & \ldots \\ G_{n1} & G_{n2} & \ldots & G_{nn} \end{pmatrix}, \quad P^*P = \begin{pmatrix} P_1^*P_1 & P_1^*P_2 & \ldots & P_1^*P_n \\ P_2^*P_1 & P_2^*P_2 & \ldots & P_2^*P_n \\ \ldots & \ldots & \ldots & \ldots \\ P_n^*P_1 & P_n^*P_2 & \ldots & P_n^*P_n \end{pmatrix}.$$

Sufficiency. Sufficiency of the theorem follows from the existence of matrices R, G_{sk} that satisfy (8). Substituting (9) into (6), we obtain (7).

Theorem 2 is proved. □

3. Volumes of a matrix ball of the second type and its skeleton

The volume of a matrix ball of the second type is calculated with the use of the following theorem [22].

Theorem 3. *Let $m \geqslant 2$ and $Z_\nu[m \times m]$ be a symmetric matrix. Let us consider the integral*

$$J(\lambda) = \int\limits_{I - \langle Z, Z\rangle > 0} [det(I - \langle Z, Z\rangle)]^\lambda \dot{Z},$$

where $\dot{Z} = \prod\limits_{i=1}^{m} \prod\limits_{j=1}^{mn} dx_{ij} dy_{ij}, x_{ij} + iy_{ij} = z_{ij}$. Then

$$J(\lambda) = \frac{\pi^{\frac{m(m+1)}{2}n}}{(\lambda+1)\dots(\lambda+mn)} \cdot \frac{\Gamma(2\lambda+3)\Gamma(2\lambda+5)\dots\Gamma(2\lambda+2mn-1)}{\Gamma(2\lambda+mn+2)\Gamma(2\lambda+mn+3)\dots\Gamma(2\lambda+2mn)}.$$

In particular, when $\lambda = 0$ the volume of a matrix ball of the second type is

$$V(B_{m,n}^{(2)}) = \frac{\pi^{\frac{m(m+1)}{2}n}}{m!} \cdot \frac{2!4!\dots(2mn-3)!}{(mn+1)!(mn+2)!\dots(2mn-1)!}. \tag{10}$$

In particular, when $n = 1$ we obtain from (10) the well-known formula for the volume of classical domain of the second type (see [8]).

Now Let us calculate the volume of the skeleton $X_{m,n}^{(2)}$ of the matrix ball of the second type $B_{m,n}^{(2)}$.

Theorem 4. *The volume of the skeleton of a matrix ball of the second type is calculated as follows*

$$V\left(X_{m,n}^{(2)}\right) = (2\pi)^{\frac{nm(m+1)}{2}} \left(\frac{D(l_1, \dots, l_m)}{1!2!\dots(m-1)! \prod\limits_{1 \leqslant s \leqslant j \leqslant m} (l_s + l_j + 2)} \right)^n,$$

where

$$D(l_1, l_2, \dots, l_m) = \prod_{1 \leqslant s < j \leqslant m} (l_s - l_j), 1 \leqslant l_k \leqslant m$$

and $l_1 + l_2 + \dots + l_m = \dfrac{m(m+1)}{2}$.

Proof. Let $U = (U_1, \dots, U_n) \in X_{m,n}^{(2)}$ and each matrix $U_k,\ k = 1, \dots, n$ is a symmetric matrix. It is known ([8, 25]) that for any symmetric matrix $Z_\nu \in \mathbb{C}[m \times m]$ there exists a unitary matrix $U_\nu \in U(m)$ ($U(m)$ are set classes of the unitary matrices group) and real numbers $\lambda_1^{(\nu)} \geqslant \lambda_2^{(\nu)} \geqslant \dots \geqslant \lambda_m^{(\nu)} \geqslant 0$ such that

$$Z_\nu = U_\nu diag\left(\lambda_1^{(\nu)}, \dots, \lambda_m^{(\nu)}\right) U'_\nu = U_\nu \Lambda_\nu U'_\nu, \Lambda_\nu = \begin{pmatrix} \lambda_1^\nu & 0 & \dots & 0 \\ 0 & \lambda_2^\nu & \dots & 0 \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & \lambda_m^\nu \end{pmatrix}, \quad \nu = 1, \dots, n. \tag{12}$$

By differentiating (12), we obtain

$$dZ_\nu = dU_\nu \Lambda_\nu U'_\nu + U_\nu d\Lambda_\nu U'{}_\nu + U_\nu \Lambda_\nu dU'{}_\nu .$$

Introducing $\delta U_\nu = U_\nu^* dU_\nu$, we have

$$U_\nu^* dZ_\nu \bar{U}_\nu = \delta U_\nu \Lambda_\nu + d\Lambda_\nu + \Lambda_\nu \delta U'{}_\nu .$$

Next we have

$$\begin{gathered} Sp\,(dZ_\nu \cdot dZ_\nu^*) = Sp\,(U_\nu^* dZ_\nu \cdot U_\nu U_\nu^* \cdot dZ_\nu^* \cdot U_\nu) = \\ = Sp\left\{(\delta U_\nu \cdot \Lambda_\nu + d\Lambda_\nu + \Lambda_\nu \delta U'{}_\nu)\left(\Lambda_\nu \delta U_\nu^* + d\Lambda_\nu + \delta \bar{U}_\nu \cdot \Lambda_\nu\right)\right\} = \\ = Sp\,(d\Lambda_\nu \cdot d\Lambda_\nu) + Sp\left\{(\delta U_\nu \cdot \Lambda_\nu + \Lambda_\nu \delta U'{}_\nu)\left(\Lambda_\nu \delta U_\nu^* + \delta \bar{U}_\nu \cdot \Lambda_\nu\right)\right\}. \end{gathered}$$

Let us set

$$\delta U_\nu \cdot \Lambda_\nu + \Lambda_\nu \delta U'{}_\nu = \left(dg_{jk}^{(\nu)}\right), \ \left(dg_{jk}^{(\nu)} = dg_{kj}^{(\nu)}\right),$$

then

$$Sp\,(dZ_\nu \cdot dZ_\nu^*) = \sum_{j=1}^{n} d\left(\lambda_j^{(\nu)}\right)^2 + \sum_{j=1}^{n} \left|dg_{jj}^{(\nu)}\right|^2 + 2\sum_{j<k}^{n} \left|dg_{jk}^{(\nu)}\right|^2,$$

where

$$\begin{gathered} dg_{jk}^{(\nu)} = \lambda_k^{(\nu)} \delta u_{jk}^{(\nu)} + \lambda_j^{(\nu)} u_{kj}^{(\nu)}, \quad j<k, \\ dg_{jj}^{(\nu)} = 2i\lambda_j^{(\nu)} \delta u_{jj}^{(\nu)}. \end{gathered}$$

Now to define the volume element $\left\{\dot{U}_\nu\right\}$ of the set $U\,(m)$ we set $\delta u_{jk}^{(\gamma)} = \delta u'{}_{jk} + i\delta u''{}_{jk}$. Then we have

$$\dot{U}_\nu = 2^{\frac{m(m-1)}{2}} \prod_{j=1}^{n} \delta u''{}_{jj} \prod_{j<k} \delta u'{}_{jk} \cdot \delta u''{}_{jk}.$$

Thus

$$\dot{Z}_\nu = 2^m \prod_{j<k} \left|\left(\lambda_j^{(\nu)}\right)^2 - \left(\lambda_k^{(\nu)}\right)^2\right| \lambda_1^{(\nu)} \ldots \lambda_m^{(\nu)} d\lambda_1^{(\nu)} \ldots d\lambda_m^{(\nu)} \dot{U}_\nu, (\lambda_j^{(\nu)} \neq \lambda_k^{(\nu)}, \nu = 1, 2, \ldots, n). \tag{13}$$

For any $Z = (Z_1, \ldots, Z_n) \in X_{m,n}^{(2)}$ we have $\det\left(I^{(m)} - \langle Z, Z\rangle\right) = 0$. On the other hand, the correspondence Z_ν and $U\,(m) \times \Lambda_\nu$ is one-to-one correspondence for all matrices

$$Z = (U_1 \Lambda_1 U'{}_1, \ldots, U_n \Lambda_n U'{}_n) \in X_{m,n}^{(2)}. \tag{14}$$

Then it follows from (12) and (14) that

$$\left(\lambda_1^{(\nu)}\right)^2 + \left(\lambda_2^{(\nu)}\right)^2 + \cdots + \left(\lambda_m^{(\nu)}\right)^2 = 1, \quad \nu = 1, 2, \ldots, n. \tag{15}$$

Then, using Fubini's theorem for calculation of the volume of the skeleton $X_{m,n}^{(2)}$, we obtain

$$V\left(X_{m,n}^{(2)}\right) = \int\limits_{X_{m,n}^{(2)}} \dot{Z} =$$

$$= 2^{mn} \int\limits_{\{U_1\}\times\{\Lambda_1\}} \prod_{j<k} \left|\left(\lambda_j^{(1)}\right)^2 - \left(\lambda_k^{(1)}\right)^2\right| \lambda_1^{(1)} \ldots \lambda_m^{(1)} d\lambda_1^{(1)} \ldots d\lambda_m^{(1)} \dot{U}_1 \times$$

$$\times \cdots \times \int\limits_{\{U_n\}\times\{\Lambda_n\}} \prod_{j<k} \left| \left(\lambda_j^{(n)}\right)^2 - \left(\lambda_k^{(n)}\right)^2 \right| \lambda_1^{(n)} \ldots \lambda_m^{(n)} d\lambda_1^{(n)} \ldots d\lambda_m^{(n)} \dot{U}_n =$$

$$= \int\limits_{\{U_1\}} \dot{U}_1 \int\limits_{\left(\lambda_1^{(1)}\right)^2+\left(\lambda_2^{(1)}\right)^2+\cdots+\left(\lambda_m^{(1)}\right)^2<1} 2^m \prod_{j<k} \left| \left(\lambda_j^{(1)}\right)^2 - \left(\lambda_k^{(1)}\right)^2 \right| \lambda_1^{(1)} \ldots \lambda_m^{(1)} d\lambda_1^{(1)} \ldots d\lambda_m^{(1)} \times$$

$$\times \cdots \times \int\limits_{\{U_n\}} \dot{U}_n \int\limits_{\left(\lambda_1^{(n)}\right)^2+\left(\lambda_2^{(n)}\right)^2+\cdots+\left(\lambda_m^{(n)}\right)^2<1} 2^m \prod_{j<k} \left| \left(\lambda_j^{(n)}\right)^2 - \left(\lambda_k^{(n)}\right)^2 \right| \lambda_1^{(n)} \ldots \lambda_m^{(n)} d\lambda_1^{(n)} \ldots d\lambda_m^{(n)}.$$

It is known (Theorem 3.1.1 in [8]) that volume of the manifold $\{U_\nu(m)\}$ of unitary matrices is calculated by the following formula

$$V\left(U_\nu(m)\right) = \frac{(2\pi)^{\frac{m(m+1)}{2}}}{1!2!\ldots(m-1)!}.$$

Providing $\lambda_1^{(\nu)} > \lambda_2^{(\nu)} > \cdots > \lambda_m^{(\nu)} > 0$ for all ν-th integral, we have

$$I_\nu = V\left(U_\nu\right) \int\limits_{\left(\lambda_1^{(\nu)}\right)^2+\left(\lambda_2^{(\nu)}\right)^2+\cdots+\left(\lambda_m^{(\nu)}\right)^2<1} 2^m \prod_{j<k} \left| \left(\lambda_j^{(\nu)}\right)^2 - \left(\lambda_k^{(\nu)}\right)^2 \right| \lambda_1^{(\nu)} \ldots \lambda_m^{(\nu)} d\lambda_1^{(\nu)} \ldots d\lambda_m^{(\nu)} =$$

$$= 2^m V\left(U_\nu\right) \int\limits_{\left(\lambda_1^{(\nu)}\right)^2+\left(\lambda_2^{(\nu)}\right)^2+\cdots+\left(\lambda_m^{(\nu)}\right)^2<1} \prod_{j<k} \left| \left(\lambda_j^{(\nu)}\right)^2 - \left(\lambda_k^{(\nu)}\right)^2 \right| \lambda_1^{(\nu)} \ldots \lambda_m^{(\nu)} d\lambda_1^{(\nu)} \ldots d\lambda_m^{(\nu)} =$$

$$= 2^m V\left(U_\nu\right) \int \cdots \int\limits_{0<\lambda_m^{(\nu)}<\cdots<\lambda_2^{(\nu)}<\lambda_1^{(\nu)}<1} \prod_{j<k} \left| \left(\lambda_j^{(\nu)}\right)^2 - \left(\lambda_k^{(\nu)}\right)^2 \right| \lambda_1^{(\nu)} \ldots \lambda_m^{(\nu)} d\lambda_1^{(\nu)} \ldots d\lambda_m^{(\nu)} =$$

$$= (-1)^{\frac{m(m-1)}{2}} V\left(U_\nu\right) \int \cdots \int\limits_{0<\lambda_m^{(\nu)}<\cdots<\lambda_2^{(\nu)}<\lambda_1^{(\nu)}<1} \det \left| \left(\lambda_s^{(\nu)}\right)^{2l_j} \right|_{s,j=1}^{m} \cdot \lambda_1^{(\nu)} \ldots \lambda_m^{(\nu)} d\lambda_1^{(\nu)} \ldots d\lambda_m^{(\nu)} =$$

$$= \frac{V\left(U_\nu\right) D\left(l_1, \ldots, l_m\right)}{\prod\limits_{1\leqslant s\leqslant j\leqslant m} \left(l_s + l_j + 2\right)},$$

where the following conditions are satisfied

$$D(l_1, l_2, \ldots, l_m) = \prod_{1\leqslant s<j\leqslant m} (l_s - l_j), \quad 1 \leqslant l_k \leqslant m$$

and $l_1 + l_2 + \cdots + l_m = \dfrac{m(m+1)}{2}$.

Here lemma from [8] (page 135) was used. Hence, we obtain relation from the statement of the theorem:

$$V\left(X_{m,n}^{(2)}\right) = (2\pi)^{\frac{nm(m+1)}{2}} \left(\frac{D\left(l_1, \ldots, l_m\right)}{1!2!\ldots(m-1)! \prod\limits_{1\leqslant s\leqslant j\leqslant m} \left(l_s + l_j + 2\right)} \right)^n.$$

Theorem 4 is proved. □

Note that when $n = 1$ we obtain the formula for calculating the volume of skeleton of classical domain of the second type (see [8]).

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О свойствах матричного шара второго типа $B_{m,n}^{(2)}$ из пространства $\mathbb{C}^n[m \times m]$

Уктам С. Рахмонов
Ташкентский государственный технический университет
Ташкент, Узбекистан
Джанибек Ш. Абдуллаев
Национальный университет Узбекистана
Ташкент, Узбекистан

Аннотация. В этой работе дано описание автоморфизмов матричного шара $B_{m,n}^{(2)}$, ассоциированных с классическими областями второго типа, также изучены некоторые свойства матричного шара второго типа.

Ключевые слова: классическая область, матричный шар, автоморфизм матричного шара, объем, границы Шилова.