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The Bergman and Cauchy–Szego Kernels for Matrix Ball of the Second Type

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With the use of holomorphic automorphism of the matrix ball of the second type the validity of the integral Bergman and Cauchy–Sege formulae is proven in this article.

Keywords: matrix ball, Bergman kernel, Cauchy–Szego kernel, automorphism of the matrix ball.

1^0. Let us assume that $\mathbb{C}[m \times m]$ is the space of complex matrices of size $[m \times m]$. Direct multiplication of n matrices is denoted by $\mathbb{C}^n[m \times m]$.

The set

$$B^{(1)}_{m,n} = \left\{ Z = (Z_1, ..., Z_n) \in \mathbb{C}^n \left[m \times m\right] : I^{(m)} - \langle Z, Z \rangle > 0 \right\}$$

is referred to as matrix ball of the first type (see [6]). Here $\langle Z, Z \rangle = Z_1 Z_1^* + Z_2 Z_2^* + ... + Z_n Z_n^*$ is the "dot" product, I is the unit matrix of size $[m \times m]$, $Z_\nu^* = \overline{Z'}_v$ is the conjugate transpose of matrix Z_ν, $\nu = 1, 2, ..., n$, and $I - \langle Z, Z \rangle > 0$ means that a Hermitian matrix is positive definite that is all matrix eigenvalues are positive.

Matrix ball the second type $B^{(2)}_{m,n}$ has the following form (see [7]):

$$B^{(2)}_{m,n} = \left\{ Z = (Z_1, ..., Z_n) \in \mathbb{C}^n \left[m \times m\right] : I^{(m)} - \langle Z, Z \rangle > 0, \;\; Z^{'}_\nu = Z_\nu, \;\; \nu = 1, ..., n \right\}.$$

Let us denote the Shilov boundary of a matrix ball $B^{(2)}_{m,n}$ by $X^{(2)}_{m,n}$, that is,

$$X^{(2)}_{m,n} = \left\{ Z \in \mathbb{C}^n \left[m \times m\right] : \langle Z, Z \rangle = I, \quad Z'_v = Z_\nu, \;\; \nu = 1, 2, ..., n \right\}.$$

This domain was originally considered in [7] and a group of holomorphic automorphisms of $B^{(2)}_{m,n}$ was described. The purpose of this paper is to find kernels of the integral Bergman and Cauchy- Szego formulae in the matrix ball of the second type. The integral Bergman formula for the matrix ball of the first type has been found in [6].

2^0. Let us consider a point $P = (P_1, P_2, ..., P_n) \in B^{(2)}_{m,n}$. Mapping

$$W_k = \overline{R}^{-1} (I^{(m)} - < Z, P >)^{-1} \sum_{s=1}^{n} (Z_s - P_s) G_{sk}, k = 1, .., n, \tag{1}$$

that transforms point P into 0 is an automorphism of the matrix ball $B^{(2)}_{m,n}$ (see [7]). Here R is a matrix of size $[m \times m]$ and G is a block matrix of size $[m \times n]$. They satisfy the following relations

$$R'(I^{(m)} - < P, P >)\overline{R} = I^{(m)}, \quad G'(I^{(mn)} - P^* P)\overline{G} = I^{(mn)}. \tag{2}$$

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Lemma 1. *Real Jacobean J_R of the mapping $W = \varphi_p(Z)$ at the point $Z = P$ is*

$$J_R\phi_P = \left(\frac{\det(I^{(m)} - < P, P >)}{\left|\det(I^{(m)} - < Z, P >)\right|^2} \right)^{\frac{(m+1)(n+1)}{2}}.$$

Proof. Let us find the real Jacobean J_R of the mapping $W = \varphi_p(Z)$ at the point $Z = P$. It follows from (1) that

$$dW_k = \overline{R}^{-1}(I^{(m)} - < Z, P >)^{-1} \sum_{i=1}^{n} dZ_i P_i^* (I^{(m)} - < Z, P >)^{-1} \sum_{s=1}^{n} (Z_s - P_s) G_{sk} +$$

$$+ \overline{R}^{-1}(I^{(m)} - < Z, P >)^{-1} \sum_{s=1}^{n} dz_s G_{sk}.$$

$$dW_k \left|_{Z=P}\right. = \overline{R}^{-1}(I^{(m)} - < Z, P >)^{-1} \sum_{s=1}^{n} dZ_s G_{sk}.$$

$$dZ \otimes G = (dZ_1, ..., dZ_n) \begin{pmatrix} G_{1k} \\ \vdots \\ G_{nk} \end{pmatrix}, k = \overline{1, n},$$

$$dW = \overline{R}^{-1}(I^{(m)} - < Z, P >)^{-1} dZ \otimes G.$$

Then we have

$$\varphi_P^{'}(P) = \overline{R}^{-1}(I^{(m)} - < Z, P >)^{-1} \otimes G,$$

where $\varphi_P^{'}$ is the Jacobi matrix of the mapping φ_P. The sign $\otimes$ means the Kronecker product of two matrices. Taking into consideration properties of the Kronecker product (see [3]) and using relation (2), we obtain

$$\det \varphi_P^{'}(P) = (\det R')^{\frac{m+1}{2}} (\det G')^{\frac{m+1}{2} n}.$$

Then applying the result of Theorem 2.1.2 from (see [2, p.37]), we find the real Jacobean of the mapping φ_Z.

Since

$$J_R \varphi_Z = \left|\det \varphi_P^{'}\right|^2$$

then

$$J_R \phi_Z(Z) = \det^{\frac{m+1}{2}}(\overline{R}R') \det^{\frac{m+1}{2} \cdot n}(\overline{G}G') = \det^{-\frac{(m+1)(n+1)}{2}}(I^{(m)} - < Z, Z >). \tag{3}$$

Taking into account relations (2), we obtain

$$\det(I^{(m)} - < W, W >) = \det(\overline{R}^{-1}(I^{(m)} - < Z, P >)^{-1}) \det(I^{(m)} - < Z, Z >) \times$$

$$\times \det((I^{(m)} - < P, Z >)^{-1} R'^{-1}) = \frac{\det(I^{(m)} - < Z, Z >)}{\det((I^{(m)} - < Z, P >)\overline{R}) \det(R'(I^{(m)} - < P, Z >))} =$$

$$= \frac{\det(I^{(m)} - < Z, Z >)}{\det(I^{(m)} - < Z, P >) \det(I^{(m)} - < P, Z >) \det(\overline{R}R')} =$$

$$= \begin{bmatrix} I^{(m)} - < P, P >= R'^{-1}\overline{R}^{-1} = (\overline{R}R')^{-1} \\ \det(I^{(m)} - < P, P >) = \det(\overline{R}R')^{-1} \\ \det(\overline{R}R') = \det^{-1}(I^{(m)} - < P, P >) \end{bmatrix} =$$

$$= \frac{\det(I^{(m)} - <Z, Z>)}{\det(I^{(m)} - <Z, P>)(\det(I^{(m)} - <Z, P>))^*(\det(I^{(m)} - <P, P>))^{-1}} =$$

$$= \frac{\det(I^{(m)} - <Z, Z>) \det(I^{(m)} - <P, P>)}{\left|\det(I^{(m)} - <Z, P>)\right|^2},$$

$$\det(I^{(m)} - <W, W>) = \frac{\det(I^{(m)} - <P, P>) \det(I^{(m)} - <Z, Z>)}{\left|\det(I^{(m)} - <Z, P>)\right|^2}. \tag{4}$$

Mapping $\psi_u = \varphi_W \circ \varphi_P \circ \varphi_Z^{-1}$ conserves 0. Therefore it is a generalized unitary mapping and the absolute value of the Jacobian determinant equals 1, i. e., $\varphi_P = \varphi_W^{-1} \circ \psi_u \circ \varphi_Z$.

Then from relations (3) and (4) we obtain

$$J_R \phi_P = \frac{\det^{\frac{(m+1)(n+1)}{2}} (I^{(m)} - <W, W>)}{\det^{\frac{(m+1)(n+1)}{2}} (I^{(m)} - <Z, Z>)} = \left(\frac{\det(I^{(m)} - <W, W>)}{\det(I^{(m)} - <Z, Z>)} \right)^{\frac{(m+1)(n+1)}{2}} =$$

$$= \left(\frac{\det(I^{(m)} - <P, P>)}{\left|\det(I^{(m)} - <Z, P>)\right|^2} \right)^{\frac{(m+1)(n+1)}{2}}. \tag{5}$$

□

3^0. Let us consider the normalized Lebesgue measures ν in the ball $B_{m,n}^{(2)}$ and σ on the Shilov boundary $X_{m,n}^{(2)}$, i.e.

$$\int_{B_{m,n}^{(2)}} d\nu(Z) = 1 \text{ and } \int_{X_{m,n}^{(2)}} d\sigma(Z) = 1.$$

Following the procedure given in [6] for $B_{m,n}^{(2)}$, the Bergman kernel is defined as follows:

$$K(Z, W) = \frac{1}{\det^{\frac{(m+1)(n+1)}{2}} (I^{(m)} - <Z, W>)}, \quad Z \in B_{m,n}^{(2)}.$$

In particular, when $n = 1$, this kernel coincides with the Bergman kernel for the classical region of the second type (see [2]).

The Hilbert space of holomorphic functions in $B_{m,n}^{(2)}$ that are square integrable with respect to Lebesgue measure $d\nu$ is designated as $H^2(B_{m,n}^{(2)})$, i.e., $f \in H^2(B_{m,n}^{(2)})$ if f is a holomorphic in $B_{m,n}^{(2)}$ fuction and

$$\int_{B_{m,n}^{(2)}} |f(\zeta)|^2 \, d\nu(\zeta) < +\infty.$$

$L^2(X_{m,n}^{(2)}, d\mu)$ signifies the space of scalar functions f that are square integrable with respect to the normalized Haar measure $d\mu$ on the Shilov boundary $X_{m,n}^{(2)}$ of the matrix ball $B_{m,n}^{(2)}$.

Theorem 1. *For each functionf $\in H^1(B_{m,n}^{(2)})$ the following relation is true*

$$f(Z) = \int_{B_{m,n}^{(2)}} f(W) K(Z, W) d\nu(W), \quad Z \in B_{m,n}^{(2)}, \; W \in X_{m,n}^{(2)}.$$

Integral in this relation defines the orthogonal projection from space $L^2(B_{m,n}^{(2)})$ to space $H^2(B_{m,n}^{(2)})$.

Proof. Let us consider a point $P \in B^{(2)}_{m,n}$. Let us assume first that the function $f \in \mathrm{A}(B^{(2)}_{m,n})$ (f is holomorphic function in $B^{(2)}_{m,n}$ and it is continuous function on the closure $\overline{B}^{(2)}_{m,n}$). Let us consider the following function

$$g(Z) = \frac{K(Z,P)}{K(P,P)} f(Z).$$

Then $g \in \mathrm{A}(B^{(2)}_{m,n})$ and

$$f(P) = g(P) = (g \circ \varphi_P^{-1})(0). \tag{6}$$

Expanding f in a series of homogeneous polynomials and integrating it over the ball, we obtain

$$f(0) = \int_{B^{(2)}_{m,n}} f(W) d\nu(W).$$

Taking into account this relation and relation (5) we have

$$f(B) = \int_{B^{(2)}_{m,n}} g(\varphi_P^{-1}(W)) d\nu(W). \tag{7}$$

After the change of variables $\varphi_P^{-1}(W) = U$ in (7), we obtain

$$f(P) = \int_{B^{(2)}_{m,n}} g(U) J_R \varphi_P d\nu(U) = \int_{B^{(2)}_{m,n}} f(U) K(P,U) d\nu(U).$$

Due to the completeness of the matrix ball the space of functions $\mathrm{A}(B^{(2)}_{m,n})$ is dense in the space $H^2(B^{(2)}_{m,n})$. Then the theorem holds for functions $f \in L^2(B^{(2)}_{m,n})$. □

4^0. Let us build the Cauchy–Szego kernel for the matrix ball of the second type.

We define the Cauchy–Szego kernel C(Z, W) as follows

$$C(Z,W) = \frac{1}{\det^{\frac{(m+1)n}{2}} (I^{(m)} - \langle Z,W \rangle)}, Z \in B^{(2)}_{m,n}, \quad W \in X^{(2)}_{m,n}. \tag{8}$$

At $n = 1$ the Cauchy–Szego formula coincides with the Cauchy–Szego kernel for the classical region of the second type [2].

This kernel is defined for all pairs $(Z,W) \in C^n[m \times m] \times C^n[m \times m]$ such that the matrix

$$I^{(m)} - \langle Z,W \rangle$$

is not degenerate matrix. In particular, the kernel is defined for $Z \in B^{(2)}_{m,n},\ W \in X^{(2)}_{m,n}$.

The kernel $C(Z,W)$ is a holomorphic function with respect to elements of the block matrix Z and it is a antiholomorphic function with respect to elements of the block matrix W.
If $f \in L^1(B^{(2)}_{m,n})$ on $X^{(2)}_{m,n}$ one can introduce the following integral

$$C[f](Z) = \int_{X^{(2)}_{m,n}} C(Z,W) f(W) d\sigma(W),\ Z \in B^{(2)}_{m,n},\ W \in X^{(2)}_{m,n}. \tag{9}$$

Let us designate $C[f]$ as Cauchy integral with respect to f. The operator that transforms f into $C[f]$ we designate as Cauchy transform.

Lemma 2. *Cauchy transform commutes with the action of the unitary group ψ_u, namely,*

$$C[f \circ \psi_u] = (C[f]) \circ \psi_u, \quad f \in L^1(\sigma).$$

Proof. Let us show that the following equality is true

$$C(Z, \psi_u^{-1} W) = C(\psi_u Z, W). \tag{10}$$

In fact, $UU^* = I^{(m)}$, $VV^* = I^{(mn)}$ are unitary and block unitary matrices. Then we have

$$C(Z, \psi_u^{-1} W) = \frac{1}{\det^{\frac{(m+1)n}{2}}(I^{(m)} - \langle Z, \psi_u^{-1} W\rangle)} = \frac{1}{\det^{\frac{(m+1)n}{2}}(I^{(m)} - \langle Z, U^{-1} W V^{-1}\rangle)} =$$

$$= \frac{1}{\det^{\frac{(m+1)n}{2}}(I^{(mn)} - Z^* \cdot U^* W V^*)} = \frac{1}{\det^{\frac{(m+1)n}{2}}(VV^* - VV^* Z \cdot U^* W V^*)} =$$

$$= \frac{1}{\det^{\frac{(m+1)n}{2}}(I^{(mn)} - (UZV)^* W)} = \frac{1}{\det^{\frac{(m+1)n}{2}}(I^{(m)} - \langle UZV, W\rangle)} = C(\psi_u Z, W).$$

Here we use the equality

$$\det(I^{(m)} - \langle Z, W\rangle) = \det(I^{(mn)} - Z^* \cdot W),$$

which is true by virtue of Theorem 2.1.2 (see [2, p. 37]) for arbitrary $Z = (Z_1, ..., Z_n)$ and $W = (W_1, ..., W_n)$. Since the measure σ is invariant with respect to ψ_u then

$$C[f \circ \psi_u] = \int_{X_{m,n}^{(2)}} C(Z, W) f(\psi_u W) d\sigma(W) = \int_{X_{m,n}^{(2)}} C(Z, \psi_u^{-1} W) f(W) d\sigma(W) =$$
$$= \int_{X_{m,n}^{(2)}} C(\psi_u Z, W) f(W) d\sigma(W) = (C[f]) \circ \psi_u.$$

□

Theorem 2. *For each functionf $\in H^1(B_{m,n}^{(2)})$ the following relation is true*

$$f(Z) = \int\limits_{X_{m,n}^{(2)}} f(W) C(Z, W) d\sigma(W), Z \in B_{m,n}^{(2)}, \quad W \in X_{m,n}^{(2)}. \tag{11}$$

Proof. Let us assume that $f \in H^1(B_{m,n}^{(2)})$ and $Z \in B_{m,n}^{(2)}$. Let us express a point $\zeta \in C^n[m \times m]$ as $\zeta = ('\zeta, \zeta_n)$, where $'\zeta = (\zeta_1, ..., \zeta_{n-1})$. By the lemma we can assume without loss of generality that $Z_n = 0$, i.e. $Z = ('Z, 0)$.

Let us introduce the following function

$$g(\zeta) = C(Z, \zeta) f(\zeta), \qquad \zeta \in B_{m,n}^{(2)}.$$

Because $Z_n = 0$ then the Cauchy–Szego kernel in $B_{m,n}^{(2)}$ coincides with the Bergman kernel $B_{m,n}^{(2)}$:

$$C(Z, \zeta) = K('Z, '\zeta).$$

Further, for any $W \in X_{m,n}^{(2)}$ function $g('W, \zeta_n)$ is the holomorphic function with respect to ζ_n in the matrix circle 5

$$W_n W_n^* - \zeta_n \zeta_n^* > 0, \tag{12}$$

and it is continuous function in the closure of the circle.

Therefore, it follows from [2, c. 91] that

$$g('W, 0) = \int_{S_n} g('W, W_n) d\sigma(W_n), \tag{13}$$

where S_n is the Shilov boundary of matrix disk (12) and $d\sigma(W_n)$ is the invariant Haar measure on S_n. Let us integrate relation (13) over $B^{(2)}_{m,n-1}$.

According to Fubini's theorem, on the right-hand side we obtain

$$\int_{X^{(2)}_{m,n}} g\sigma(W) = C[f](Z).$$

Because $g('W, 0) = K(Z,' W)f('W, 0)$ then it follows from Theorem 1 that the integral on the left-hand side of (13) is $f('Z, 0) = f(Z)$.
The theorem is proved. □

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Ядра Бергмана и Коши-Сеге для матричного шара второго типа

Гулмирза Х. Худайберганов
Уктам С. Рахмонов

С помощью голоморфности автоморфизмов матричного шара второго типа доказана справедливость интегральных формул Бергмана и Коши–Сеге.

Ключевые слова: матричный шар, ядро Бергмана, ядро Коши-Сеге, автоморфизм матричного шара.