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Sharp Theorems on Traces in Analytic Spaces in Tube Domains over Symmetric Cones

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New sharp estimates of Traces in Bergman-type spaces of analytic functions on tube domains over symmetric cones are obtained.These are first results of this type for tube domains over symmetric cones.

Keywords: trace estimates, tube domains, analytic functions, Bergman-type spaces.

1. Introduction and preliminaries

In this note we obtain new sharp estimates for Traces in Bergman type spaces of analytic spaces in tube domains over symmetric cones. This line of investigation can be considered as a continuation of our previous papers on Traces in analytic function spaces [1,3] and [2,4–6] where similar results were obtained but only in bounded domains in higher dimension. We remark that in this note for the first time in literature we consider this known problem related with Trace estimates in spaces of analytic functions in unbounded domains in C^n, namely in tube domains over symmetric cones. The first section contains required preliminaries on analysis on symmetric cones. Our new sharp results are contained in the second section of this note. The Whitney type decomposition of tubular domain based on properties of so-called Bergman balls (see [7–9]) serves as important tool for almost all our proofs as in previous papers on this problem in other domains [2–4]. In one dimensional tubular domain which is upperhalfspace C_+ (see [7]) our theorems are not new and they were obtained recently in [6]. Moreover arguments in proofs are parallel to those we have in one dimension or polyball in C^n The base of proof is again the so-called Bergman reproducing formula, but in tubular domain over symmetric cone(see, for example, [7–9] for this integral representation). We shortly remind the history of Diagonal map (or Trace) problem. After the appearance of [10] various papers appeared where arguments which can be seen in [10] were extended, changed and modified in various directions in one and higher dimension (see, for example, [11–15] and [1,3,4] and also various references there).

In particular in mentioned papers various new sharp results on traces for analytic function spaces in higher dimension (unit polyballball) were obtained. New results for large scales of analytic Q_p type spaces in polyball were proved(see [6]).

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Later several new sharp results for harmonic functions of several variables in the unit ball and upperhalfplane of Euclidean space were also obtained (see, for example, [1] and references there).

Probably for the first time in literature these type problems connected with diagonal map in analytic spaces appeared before in [10].

In this book this problem was formulated and certain cases connected with spaces of analytic functions in the unit disk were considered.

These results and various other results were mentioned and proved much later in [14]. Some interesting applications of diagonal map can be seen in [14,16] where other problems around this topic can be found also. The goal of this note to develop further some ideas from our recent mentioned papers and present a new sharp theorems in tube domain over symmetric cones. In upper halfplane of complex plane C_+ which is a tube domain in one dimension such results already were obtained previously by author [2], so this problem in tube appears naturally. For formulation of our results we will need various standard definitions from the theory of tube domains over symmetric cones(see [7,17–19]). In this section we also mention some vital facts which will be heavily used in proofs of our assertions (see, for example, also for parallel assertion in other domains [2–5]).

Let $T_\Omega = V + i\Omega$ be the tube domain over an irreducible symmetric cone Ω in the complexification $V^{\mathbb{C}}$ of an n-dimensional Euclidean space $\widetilde{V}$. Following the notation of [17] and [7] we denote the rank of the cone Ω by r and by Δ the determinant function on $\widetilde{V}$. Letting $\widetilde{V} = \mathbb{R}^n$, we have as an example of a symmetric cone on $\mathbb{R}^n$ the forward light cone Λ_n defined for $n \geqslant 3$ by

$$\Lambda_n = \{y \in \mathbb{R}^n : y_1^2 - \cdots - y_n^2 > 0, y_1 > 0\}.$$

Light cones have rank 2. The determinant function in this case is given by the Lorentz form

$$\Delta(y) = y_1^2 - \cdots - y_n^2.$$

(see, for example , [7]).

Let us introduce some convinient notations regarding multi-indicses.

If $t = (t_1, \dots, t_m)$, then $t^\star = (t_m, \dots, t_1)$ and, for $a \in \mathbb{R}$, $t + a = (t_1 + a, \dots, t_m + a)$. Also, if $t, k \in \mathbb{R}^m$, then $t < k$ means $t_j < k_j$ for all $1 \leqslant j \leqslant m$.

We are going to use the following multi-index

$$g_0 = \left((j-1)\frac{d}{2}\right)_{1\leqslant j\leqslant r}, \quad \text{where} \quad (r-1)\frac{d}{2} = \frac{n}{r} - 1.$$

$\mathcal{H}(T_\Omega)$ denotes the space of all holomorphic functions on T_Ω. We denote m cartesian products of tubes by T_Ω^m, $T_\Omega^m = T_\Omega \times ... \times T_\Omega$ the space of all analytic function on this new product domain which are analytic by each variable separately will be denoted by$\mathcal{H}(T_\Omega^m)$. In this paper we will be interested on properties of certain analytic subspaces of $\mathcal{H}(T_\Omega^m)$. By m we denote below a natural number bigger than 1. For $\tau \in \mathbb{R}_+$ and the associated determinant function $\Delta(x)$ we set

$$A_\tau^\infty(T_\Omega) = \left\{F \in \mathcal{H}(T_\Omega) : \|F\|_{A_\tau^\infty} = \sup_{x+iy\in T_\Omega} |F(x+iy)|\Delta^\tau(y) < \infty\right\}, \tag{1}$$

([7] and references there). It can be checked that this is a Banach space.

For $1 \leqslant p,\ q < +\infty$ and $\nu \in \mathbb{R}$, and $\nu > \dfrac{n}{r} - 1$ we denote by $A_\nu^{p,q}(T_\Omega)$ the mixed-norm weighted Bergman space consisting of analytic functions f in T_Ω that

$$\|F\|_{A_\nu^{p,q}} = \left(\int_\Omega \left(\int_{\tilde{V}} |F(x+iy)|^p dx\right)^{q/p} \Delta^\nu(y)\frac{dy}{\Delta(y)^{n/r}}\right)^{1/q} < \infty.$$

This is a Banach space. Replacing above simply A by L we will get as usual the corresponding larger space of all measurable functions in tube over symmetric cone with the same quazinorm (see [17,19]). It is known the $A_\nu^{p,q}(T_\Omega)$ space is nontrivial if and only if $\nu > \frac{n}{r} - 1$,(see [7,18]). And we will assume this everywhere below. When $p = q$ we write (see [7])

$$A_\nu^{p,q}(T_\Omega) = A_\nu^p(T_\Omega).$$

This is the classical weighted Bergman space with usual modification when $p = \infty$.

Let $T_\Omega^m = T_\Omega \times ... \times T_\Omega$. To define related two Bergman-type spaces $A_\nu^p(T_\Omega^m)$ and $A_\tau^\infty(T_\Omega^m)$ (ν and τ can be also vectors) in m-products of tube domains T_Ω^m we follow standard procedure which is well-known in case of unit disk and unit ball [2–4,10,14]. Namely we consider analytic F functions $F = F(z_1, \dots z_m)$ which are analytic by each $z_j, j = 1, \dots, m$ variable, and where each such variable belongs to T_Ω tube and define as $H(T_\Omega^m)$ the space of all such functions. For example we set, for all $z_j = x_j + y_j$, $\tau_j \in R$, $j = 1, \dots, m$, $F(z) = F(z_1, \dots, z_m)$, $\tau = (\tau_1, \dots, \tau_m)$

$$A_\tau^\infty(T_\Omega^m) = \left\{ F \in \mathcal{H}(T_\Omega^m) : \|F\|_{A_\tau^\infty} = \sup_{x+iy \in T_\Omega^m} |F(x+iy)|\Delta^\tau(y) < \infty \right\}, \tag{2}$$

$$|F(x+iy)| = |F(x_1 + iy_1, ..., x_m + iy_m)|,$$

where $\Delta^\tau(y)$ is a product of m-onedimensional $\Delta^{\tau_j}(y_j)$ functions, $j = 1, \dots, m$. Similarly the Bergman space A_τ^p can be defined on products of tubes for all $\tau = (\tau_1, \dots, \tau_m)$, $\tau_j > \frac{n}{r} - 1, j = 1, ..., m$. It can be shown that all spaces are Banach spaces. Replacing above simply A by L we will get as usual the corresponding larger space of all measurable functions in products of tubes over symmetric cone with the same quazinorm.

The (weighted) Bergman projection P_ν is the orthogonal projection from the Hilbert space $L_\nu^2(T_\Omega)$ onto its closed subspace $A_\nu^2(T_\Omega)$ and it is given by the following integral formula (see [7])

$$P_\nu f(z) = C_\nu \int_{T_\Omega} B_\nu(z,w) f(w) dV_\nu(w), \tag{3}$$

where

$$B_\nu(z,w) = C_\nu \Delta^{-(\nu + \frac{n}{r})}((z - \overline{w})/i)$$

is the Bergman reproducing kernel for

$$A_\nu^2(T_\Omega)$$

(see [7,17]).

Here we used the notation

$$dV_\nu(w) = \Delta^{\nu - \frac{n}{r}}(v) du dv.$$

Below and here we use constantly the following notations $w = u + iv \in T_\Omega$ and also $z = x + iy \in T_\Omega$.

Hence for any analytic function from $A_\nu^2(T_\Omega)$ the following reproducing integral formula is valid(see also [7])

$$f(z) = C_\nu \int_{T_\Omega} B_\nu(z,w) f(w) dV_\nu(w). \tag{4}$$

We will use even more general version of this assertion for $A_\nu^{p,q}$. We denote everywhere below by C_β the Bergman representation constant.

Reproducing formulas are bases of all our proofs as in simpler cases of unit disk and unit polyball. In this case we say simply that the f function allows the Bergman representation via Bergman kernel with ν index.

Note these assertions have direct copies in simpler cases of analytic function spaces in unit disk, polydisk, unit ball, upperhalfspace C_+ and in spaces of harmonic functions in the unit ball or upperhalfspace of the Euclidean space R^n. These classical facts are well-known and can be found, for example, in [14] and in some items from references there.

Above and throughout the paper we write C (sometimes with indexes) to denote positive constants which might be different each time we see them (and even in a chain of inequalities), but is independent of the functions or variables being discussed.

In this paper we will also need a pointwise estimate for the Bergman projection of functions in $L^{p,q}(T_\Omega)$, defined by integral formula ([7]), when this projection makes sense. Note such estimates in simpler cases of unit disk, unit ball and polydisk are well-known (see [13]).

Let us first recall the following known basic integrability properties for the determinant function, which appeared already above in definitions. Below we denote by Δ_s the generalized power function [7, 17].

Lemma 1. *1) The integral*

$$J_\alpha(y) = \int_{\mathbb{R}^n} \left| \Delta^{-\alpha} \left(\frac{x+iy}{i} \right) \right| dx$$

converges if and only if $\alpha > 2\dfrac{n}{r} - 1$. *In that case*

$$J_\alpha(y) = \widetilde{C}_\alpha \Delta^{-\alpha+n/r}(y),$$

$\alpha \in R,\ y \in \Omega.$

2)Let $\alpha \in \mathbb{C}^r$ *and* $y \in \Omega$. *For any multi-indices* s *and* β *and* $t \in \Omega$ *the function*

$$y \mapsto \Delta_\beta(y+t)\Delta_s(y)$$

belongs to $L^1(\Omega, \frac{dy}{\Delta^{n/r}(y)})$ *if and only if* $\Re s > g_0$ *and* $\Re(s+\beta) < -g_0^*$. *In that case we have*

$$\int_\Omega \Delta_\beta(y+t)\Delta_s(y)\frac{dy}{\Delta^{n/r}(y)} = \widetilde{C}_{\beta,s}\Delta_{s+\beta}(t).$$

We refer to Corollary 2.18 and Corollary 2.19 of [18] for the proof of the above lemma or [7]. As a corollary of one dimensional version of second estimate and first estimate (see, for example, [19] Theorem 3.9) we obtain the following vital Forelly-Rudin type estimate for Bergman kernel (A) which we will use in proof of our main result [19]

Lemma 2.

$$\int_{T_\Omega} \Delta^\beta(y)|B_{\alpha+\beta+\frac{n}{r}}(z,w)|dV(z) \leqslant C\Delta^{-\alpha}(v), \tag{5}$$

$\beta > -1,\ \alpha > \dfrac{n}{r} - 1,\ z = x+iy,\ w = u+iv$ *(see [19]).*

We will also need for our proofs the following important fact on integral representations (see [9]).

Lemma 3. *Let* $\nu > \dfrac{n}{r} - 1$, $\alpha > \dfrac{n}{r} - 1$, *then for all functions from* A_α^∞ *the integral representations of Bergman with Bergman kernel* $B_{\alpha+\nu}(z,w)$ *(with* $\alpha+\nu$ *index) is valid.*

The following result can be found in [19](section4).

Lemma 4. *For all* $1 < p < \infty$ *and* $1 < q < \infty$ *and for all* $\dfrac{n}{r} \leqslant p_1$, *where* $\dfrac{1}{p_1} + \dfrac{1}{p} = 1$ *and* $\dfrac{n}{r} - 1 < \nu$ *and for all functions* f *from* $A_\nu^{p,q}$ *and for all* $\dfrac{n}{r} - 1 < \alpha$ *the Bergman representation formula with* α *index or with the Bergman kernel* $B_\alpha(z,w)$ *is valid.*

We remark this result is a particular case of a more general assertion for analytic mixed norm $A^{p,q}_\nu$ classes (see [19]), which means that our main result below partially admits also some extentions, even to mixed norm spaces which we defined above.

We note also that (see [19])

$$|f(x+iy)|\Delta^{\frac{n}{rp}+\frac{\nu}{q}}(y) \leqslant c_{p,q,r,\nu}\|f\|_{A^{p,q}_\nu}, \quad 1 \leqslant p,q < \infty, \nu > \frac{n}{r}-1. \tag{6}$$

All the mentioned results together with properties of the Whitney decomposition of tubular domain over symmetric cones based on Bergman balls [7,8] are used heavily during all proofs of our assertions.

Lemma 5 ([7–9]). *Given $\delta \in (0;1)$ there exists a sequence of points $\{z_j\}$ in T_Ω called δ-lattice such that calling $\{B_j\}$ and $\{B'_j\}$ the Bergman balls with center z_j and radius δ and $\delta/2$ respectively then*

A) the balls $\{B'_j\}$ are pairwise disjoint;

B) the balls $\{B_j\}$ cover T_Ω with finite overlapping;

C) $\displaystyle\int_{B_j(z_j,\delta)} \Delta^s(y)dV(z) \asymp \int_{B'_j(z_j,\delta)} \Delta^s(y)dV(z) = \widetilde{C}_\delta \Delta^{2\frac{n}{r}+s}(\operatorname{Im} z_j);$

$s > \dfrac{n}{r}-1, J = |B_\delta(z_j)| \asymp \Delta^{\frac{2n}{r}}(\operatorname{Im} z_j), j = 1,...,m, J \asymp \Delta^{\frac{2n}{r}}(\operatorname{Im} w), w \in B_\delta(z_j).$

Lemma 6 ([7–9]). *For any $f \in A^2_\nu(T_\Omega)$ we have for any $\mu > \dfrac{2n}{r}-1$*

$$f(z) = \int_{T_\Omega} B_\mu(z,w)f(w)\Delta^{\mu-\frac{2n}{r}}(\operatorname{Im} w)dV(w).$$

Lemma 7 (Besov space, [7–9]). *Let $\Box_z$ be the natural extension to the complex plane space C^n of the generalized wave operator $\Box_x$ on the cone Ω $\Box_z = [\Delta(\frac{1}{i}\frac{\partial}{\partial z})]$ which is the differential operator of degree r. We define for $1 \leq p < \infty$ as $B^p(T_\Omega)$ the Besov space the sincee of all holomorphic functions f so that*

$$\int_{T_\Omega} |\Box^n f(x+iey)|^p \Delta^{np-\frac{2n}{r}}(y)dxdy < \infty.$$

$B^p_\nu = A^p_\nu,\ \nu > \dfrac{n}{r}-1,\ 1 \leqslant p < \overline{p}_\nu,\quad A^p_\nu \subset B^p_\nu,\ \nu > \dfrac{n}{r}-1,$

$1 \leqslant p < \widetilde{p}_\nu,\quad \overline{p} = \dfrac{\nu+\frac{2n}{r}-2}{\frac{n}{r}-1},\ \ \widetilde{p} = \dfrac{\nu+\frac{2n}{r}-1}{\frac{n}{r}-1}.$

Lemma 8 ([7–9]). *a) Let $\nu > \dfrac{n}{r}-1,\ 1 \leqslant p \leqslant \infty$. For all $F \in A^p_\nu$ we have*

$$\Box^l F(z) = C\int_{T_\Omega} B_{\nu+l}(z,w)\Box^m F(w)\Delta^m(\operatorname{Im} w)dV_\nu(w)$$

$m \geqslant 0$, l is large enough;

For $l = 0$ this is valid, $1 \leqslant p < \widetilde{R}$, for any $\widetilde{R} > 1$.

b) Let $\nu > \frac{n}{r}-1, \alpha > \frac{n}{r}-1$. Then for all $\Delta^\alpha(\operatorname{Im} z)F(z) \in L^\infty$ and all $m \geqslant 0$:

$$F(z) = C\int_{T_\Omega} B_{\nu+\alpha}(z,w)\Box^m F(w)\Delta^{m+\alpha}(\operatorname{Im} w)dV_\nu(w).$$

2. On sharp estimates for traces in analytic function spaces in tube domains over symmetric cones

In this paper we restrict ourselves to Ω irreducible symmetric cone in the Euclidean vector space R^n of dimension n, endowed with an inner product for which the cone Ω is self dual. We denote by $T_\Omega = R^n + i\Omega$ the corresponding tube domain in C^n.

This section is devoted to formulations and proofs of all main results of this paper. As previously in case of analytic functions in unit disk, polydisk, unit ball and upperhalfspace C_+ and in case of spaces of harmonic functions in Euclidean space [1, 3, 4] the role of the Bergman representation formula is crucial in these issues and our proofs are heavily based on it and some lemmas we provided above and they are parallel to cases we considered before [2–4].

As it is known a variant of Bergman representation formula is available also in Bergman-type analytic function spaces in tubular domains over symmetric connes and this known fact (see [7, 17–19]), which is crucial also in various other problems in analytic function spaces in tubular domains (see [7] and various references there) is used also in our proofs below.

Theorem 1. *Let $f \in A^p_\nu(T^m_\Omega)$, $1 \leqslant p < \infty$, $\nu \in R^n$, $\nu_j > \nu_0$ for fixed $\nu_0 = \nu_0(p, n, r, m)$, for all $j = 1, \dots, m$. Then $f(z, \dots, z) \in A^p_s$, where $s = \sum_{j=1}^{m}(\nu_j - \frac{n}{r}) + 2\frac{n}{r}(m-1)$ with related estimates for norms. And for all $\frac{n}{r} \leqslant p_1$, where p_1 is a conjugate of p the reverse is also true. For each g function $g \in A^p_s(T_\Omega)$ there is an F function, $F(z, \dots, z) = g(z)$, $F \in A^p_\nu(T^m_\Omega)$.*

Let in addition

$$\Big(T_\beta f\Big)(z_1, \dots, z_m) = C_\beta \int_{T_\Omega} f(w) \prod_{j=1}^{m} \Delta^{-t}((z_j - \overline{w})/i) dV_\beta(w),$$

$mt = \beta + \dfrac{n}{r}$, $z_j \in T_\Omega$, $j = 1, \dots, m$.

Then the following asertions hold for all β so that $\beta > \beta_0$ for some fixed large enough positive number β_0. The T_β Bergman-type integral operator (expanded Bergman projection) maps $A^p_s(T_\Omega)$ to $A^p_\nu(T^m_\Omega)$, $\nu = (\nu_1, \dots, \nu_m), \nu_j > \nu_0, j = 1, ..., m$.

Proof. We have using A), B), C) of lemma 5 the following chain of estimates for family $B_\delta(z_j)$ of Bergman balls, $z_j = x_j + iy_j$

$$I = \int_{T_\Omega} |F(z, ..., z)|^p \Delta^{\sum_{j=1}^{m}(\nu_j - \frac{n}{r}) + 2\frac{n}{r}m - 2\frac{n}{r}}(y) dydx = (B_j = B_\delta(z_j)) = B_j(z_j, \delta) =$$

$$= \sum_{j=1}^{\infty} \int_{B_j} |F(z, ..., z)|^p \Delta^{\sum_{j=1}^{m}(\nu_j - \frac{n}{r}) + 2\frac{n}{r}(m-1)}(y) dydx \leqslant \widetilde{C} \sum_{j=1}^{\infty} \left(\sup_{z \in B_j} |F(z, ..., z)|^p \right) \left[\int_{B_j} \Delta^\tau(y) dydx \right].$$

By lemma 5

$$I = \int_{T_\Omega} |F(z, ..., z)|^p \Delta^{\sum_{j=1}^{m}(\nu_j - \frac{n}{r}) + 2\frac{n}{r}m - 2\frac{n}{r}}(y) dydx = (B_j = B_\delta(z_j)) = B_j(z_j, \delta) =$$

$$= \sum_{j=1}^{\infty} \int_{B_j} |F(z, ..., z)|^p \Delta^{\sum_{j=1}^{m}(\nu_j - \frac{n}{r}) + 2\frac{n}{r}(m-1)}(y) dydx \leqslant \widetilde{C} \sum_{j=1}^{\infty} \left(\sup_{z \in B_j} |F(z, ..., z)|^p \right) \left[\int_{B_j} \Delta^\tau(y) dydx \right].$$

By lemma 5

$$\leqslant \widetilde{C}_1 \sum_{j_1=1}^{\infty} ... \sum_{j_m=1}^{\infty} \sup_{\substack{z_1 \in B_{j_1} \\ \dots \\ z_m \in B_{j_m}}} |F(z_1, ..., z_m)|^p \Big[\Delta^{\nu_1 + \frac{n}{r}}(\mathrm{Im} z_1)\Big] ... \Big[\Delta^{\nu_m + \frac{n}{r}}(\mathrm{Im} z_m)\Big] \leqslant$$

$$\leqslant \widetilde{C}_2 \sum_{j_1=1}^{\infty} ... \sum_{j_m=1}^{\infty} \int_{(B_{j_1})'} ... \int_{(B_{j_m})'} |F(z_1, ..., z_m)|^p \Delta^{\nu_1 - \frac{n}{r}}(\mathrm{Im}\, z_1) .... \Delta^{\nu_m - \frac{n}{r}}(\mathrm{Im}\, z_m) \times$$

$$\times dx_1 ... dx_m dy_1 ... dy_m = \int_{T_\Omega} ... \int_{T_\Omega} |F(z_1, ..., z_m)|^p \prod_{j=1}^{m} \Delta^{\nu_j - \frac{n}{r}}(\mathrm{Im}\, z_j) dz_1 ... dz_m;$$

We used the fact that for $1 \leqslant p < +\infty, B_j = B_\delta(z_j) = B_j(z_j, \delta), j = 1, ..., m, \ \delta > 0$

$$|F(z_j)|^p \leqslant C\left[\int_{B'_\delta(z_j)} \frac{|F(w)|^p dudv}{\Delta^{2\frac{n}{r}}(v)}\right] \asymp \frac{C}{|B_\delta(z_j)|}\int_{B'_\delta(z_j)} |F(w)|^p dudv, z_j \in T_\Omega; \tag{7}$$

(see [7–9]) m-times by each variable $z_1, ..., z_m$ and item (7) of lemma 5 at last step.

It remains to show the second part of this theorem. Let

$$T_\beta[f(z_1, ..., z_m)] = C_\beta \int_{T_\Omega} f(w)\Big[\prod_{j=1}^m \Delta^{-t}\Big(\frac{z_j - \overline{w}}{i}\Big)\Big] dV_\beta(w), \ mt = \Big(\beta + \frac{n}{r}\Big); \ z_j \in T_\Omega, \ j = 1, ..., m;$$

β is large enough. Then $\Big(T_\beta f\Big)(z, ..., z) = f(z); \ z \in T_\Omega, \ f \in A^p_s; \ \frac{n}{r} \leqslant p_1; \ s > \frac{n}{r} - 1, \ 1 \leqslant p < +\infty$ by lemma 4. We show now that T_β acts from (A^p_s) to (A^p_ν), if β is large enough.

This will finish the proof of our theorem.

For this we will use lemma 2.

We have the following chain of estimates for T_β integral operator (expanded Bergman projector).

First we have using Holder's inequality twice and lemma 2 $\gamma_1 + \gamma_2 = t; \ \frac{1}{p} + \frac{1}{p_1} = 1; \ \beta > \frac{n}{r} - 1; \ \gamma_2 > \Big(2\frac{n}{r} - 1 + \beta\Big)\left(\frac{1}{p_1 m}\right); \ \gamma_2 < \Big[\min \nu_j - \frac{n}{r} + \Big(\beta + \frac{n}{r}\Big)\left(\frac{p}{p_1 m}\right) + 1\Big]\frac{1}{p};$

$$\left(\int_{T_\Omega} |f(w)| \prod_{j=1}^m |\Delta^{-t}\Big(\frac{z_j - \overline{w}}{i}\Big)| dV_\beta(w)\right)^p \leqslant C(I \cdot J) =$$

$$= \widetilde{C}\left(\int_{T_\Omega} \frac{|f(w)|^p dV_\beta(w)}{\prod_{j=1}^m |\Delta^{\gamma_1 p}\Big(\frac{z_j - \overline{w}}{i}\Big)|}\right)\left(\int_{T_\Omega} \frac{dV_\beta(w)}{\prod_{j=1}^m |\Delta^{\gamma_2 p_1}\Big(\frac{z_j - \overline{w}}{i}\Big)|}\right)^{\frac{p}{p_1}}, \ z_j \in T_\Omega, j = 1, ..., m.$$

$$J \leqslant C\prod_{j=1}^m \left(\int_{T_\Omega} \frac{dV_\beta(w)}{|\Delta^{\gamma_2 p_1 m}\Big(\frac{z_j - \overline{w}}{i}\Big)|}\right)^{\frac{p}{p_1 m}} \leqslant \widetilde{C}_1\left(\prod_{j=1}^m \Delta^{-\tau}(\operatorname{Im} z_j)\right), \ z_j \in T_\Omega, j = 1, ..., m;$$

for $m\left(\frac{p_1}{p}\right)\tau = (\gamma_2 p_1 m) - \beta - \frac{n}{r}; \ \gamma_2 > \frac{\beta + 2\frac{n}{r} - 1}{p_1 m}; \ \beta > \frac{n}{r} - 1$ and hence we have now using this the following estimate

$$\int_{T_\Omega} ... \int_{T_\Omega} |(T_\beta f)(z_1, ..., z_m)|^p \Big(\Delta^{\nu_1 - \frac{n}{r}}(\operatorname{Im} z_1)\Big) ... \Big(\Delta^{\nu_m - \frac{n}{r}}(\operatorname{Im} z_m)\Big)\prod_{j=1}^m dx_j dy_j \leqslant \widetilde{C}\|f\|^p_{A^p_s},$$

$z_j = x_j + iy_j, j = 1, ..., m.$

The last estimate follows again directly from inequality of lemma 2 and Fubini's theorem and some calculations based on estimate

$$\int_{T_\Omega} ... \int_{T_\Omega} \frac{\Big[\Delta^{\nu_1 - \frac{n}{r} - \tau}(\operatorname{Im} z_1)\Big] ... \Big[\Delta^{\nu_m - \frac{n}{r} - \tau}(\operatorname{Im} z_m)\Big]\prod_{j=1}^m dx_j dy_j}{\prod_{j=1}^m |\Delta^{\gamma_1 p}\Big(\frac{z_j - \overline{w}}{i}\Big)|} \leqslant \widetilde{C}_3\Big[\Delta^{v}(\operatorname{Im} w)\Big], w \in T_\Omega;$$

where $\nu_j - \frac{n}{r} - \tau > -1$; $j = 1, ..., m$; $v = \sum_{j=1}^{m}(\nu_j - \frac{n}{r}) + 2\frac{n}{r}(m-1) + \frac{n}{r} - \beta$; $-v > \frac{n}{r} - 1$ and β is large enough.

The proof of theorem 1 is now complete. □

A complete analogue of this theorem is true also for $p = \infty$ case. This second theorem follows directly from Bergman reproducing formula provided above for A^∞_ν and some elementary estimates like Holder inequality for m functions and estimate (5).

Theorem 2. *Let $f \in A^\infty_\nu(T^m_\Omega)$, $\nu \in R^n$, $\nu_j > \frac{n}{r} - 1$, for all $j = 1, \dots, m$. Then $f(z, \dots, z) \in A^\infty_s$, where $s = \sum_{j=1}^{m} \nu_j$. And the reverse is also true — for each g function $g \in A^\infty_s(T_\Omega)$ there is an F function, $F(z, \dots, z) = g(z)$, $F \in A^\infty_\nu(T^m_\Omega)$. Let in addition*

$$\Big(T_\beta f\Big)(z_1, \dots, z_m) = C_\beta \int_{T_\Omega} f(w) \prod_{j=1}^{m} \Delta^{-t}((z_j - \overline{w})/i) dV_\beta(w),$$

$mt = \beta + \frac{n}{r}$, $z_j \in T_\Omega, j = 1, ..., m$. Then the following assertions hold for all β, $\beta > \beta_0$ for some fixed large enough positive number β_0.

The T_β Bergman-type integral operator (expanded Bergman projection) maps $A^\infty_s(T_\Omega)$ to $A^\infty_\nu(T^m_\Omega)$, $\nu = (\nu_1, \dots, \nu_m)$, $\nu_j > \frac{n}{r} - 1$, $j = 1, ..., m$, $s = \sum_{j=1}^{m} \nu_j$.

Proof. Note using the obvious property of $\Delta^\tau(y)$ function we have one part of theorem since we have obviously that

$$\sup_{z \in T_\Omega} |f(z, ..., z)| [\Delta^\tau(y)] \leqslant \sup_{z_1 \in T_\Omega} ... \sup_{z_m \in T_\Omega} |f(z_1, ..., z_m)| [\Delta^{\tau_1}(\operatorname{Im} z_1)] ... \times$$

$$\times [\Delta^{\tau_m}(\operatorname{Im} z_m)]; \tau_1 + ... + \tau_m = \tau > 0, \quad \tau_j > \frac{n}{r} - 1, j = 1, ..., m, \quad z = x + iy.$$

Let us show the reverse implication for this theorem. For this we have to use heavily two lemmas which we formulated above, namely lemma 3 and lemma 2.

First we have that if $f \in A^\infty_s(T_\Omega)$; $s = \sum_{j=1}^{m} \nu_j$ and if

$$\Big(T_\beta f\Big)(z_1, ..., z_m) = C_\beta \int_{T_\Omega} f(w) \Big[\prod_{j=1}^{m} \Delta^{-t}\Big(\frac{z_j - \overline{w}}{i}\Big)\Big] dV_\beta(w), mt = \beta + \frac{n}{r}; \beta > \beta_0, z_j \in T_\Omega$$

then $\Big(T_\beta f\Big)(z, ..., z) = f(z)$, $z \in T_\Omega$ by lemma 3 for all β, $\beta > \beta_0$, $z_j \in T_\Omega$, $j = 1, ..., m$.

Then we have that by Holder's inequality for m functions and (5), $\beta_1 = \beta - s$

$$|\Big(T_\beta f\Big)(z_1, ..., z_m)| [\Delta^{\nu_1}(\operatorname{Im} z_1) ... \Delta^{\nu_m}(\operatorname{Im} z_m)] \leqslant C_\beta \|f\|_{A^\infty_s} \times$$

$$\times \left(\int_{T_\Omega} \prod_{j=1}^{m} \Delta^{-t}\Big(\frac{z_j - \overline{w}}{i}\Big) dV_{\beta_1}(w) \right) \left[\prod_{j=1}^{m} \Delta^{\nu_j}(\operatorname{Im} z_j) \right] \leqslant C_\beta \|f\|_{A^\infty_s}.$$

$$\left[\int_{T_\Omega} \prod_{j=1}^{m} \Delta^{-t}\Big(\frac{z_j - \overline{w}}{i}\Big) \Big] dV_{\beta_1}(w) \right] \prod_{j=1}^{m} \Delta^{\nu_j}(\operatorname{Im} z_j) \leqslant$$

$$\leqslant C\prod_{j=1}^{m}\left(\int_{T_\Omega}\frac{\Delta^{\beta-\frac{n}{r}-\nu_j m}(\operatorname{Im} w)dudv}{\Delta^{\beta+\frac{n}{r}}\left(\frac{z_j-\overline{w}}{i}\right)}\right)^{\frac{1}{m}}\left[\prod_{j=1}^{m}\Delta^{\nu_j}(\operatorname{Im} z_j)\right],\quad \|f\|_{A_s^\infty}<\infty.$$

for all $\beta>\beta_0$ and $\nu_j>\dfrac{n}{r}-1, w=u+iv, j=1,...,m.$

Hence we have

$$|\Big(T_\beta f\Big)(z_1,...,z_m)|\prod_{j=1}^{m}\Delta^{\nu_j}(\operatorname{Im} z_j)\leqslant \widetilde{C}_\beta\|f\|_{A_s^\infty}, z_j\in T_\Omega, j=1,...,m.$$

The proof of theorem 2 is complete. □

Let

$$L_\tau^p(T_\Omega^m)=\{f-\ \text{loc. integrable in } T_\Omega:$$

$$\|f\|_{L_\tau^p}=\int_{T_\Omega}|f(z)|^p\prod_{j=1}^{m}\Delta^{\tau_j-n/r}(y_j)dx_jdy_j<\infty\}.$$

To define the next space of functions we remind the reader that the family of Bergman balls $B_\delta(z)$ forms an r-lattice in tubular domain T_Ω ([7–9]). We denote by $B_\delta^m(Z)$ standard m-cartesian product of such Bergman balls in C^m, $Z=(z_1,\dots,z_m)$. By $K_{\nu,\tau}^p(T_\Omega^m)$ we denote all f analytic functions in T_Ω^m, so that

$$\int_{B_\delta^m(Z)}|f(z_1,\dots,z_m)|^p\prod_{j=1}^{m}\Delta^{s_j}(y_j)dx_jdy_j$$

belongs to $L^1_{\tau_1,\dots,\tau_m}(T_\Omega^m)$, where $s_j=\nu_j-\dfrac{n}{r}$ for all $j=1,\dots,m$, and where $1\le p<\infty$, $\nu_j>\dfrac{n}{r}-1,\ \tau_j>\dfrac{n}{r}-1$, for all $j=1,\dots,m$. Note in polyball complete analogues of these classes were considered in [3] and the complete description of Traces of these spaces were also given. We obtain below a complete analogue of that result in case of tubular domain over symmetric cone.

Theorem 3. *Let $\nu_j>\nu_0$ and $\tau_j>\tau_0$ for some fixed positive numbers $\nu_0=\nu_0(p,n,r,m)$ and $\tau_0=\tau_0(p,n,r,m)$, $1\leqslant p<\infty$. Let $f\in K_{\nu,\tau}^p(T_\Omega^m)$ then $f(z,\dots,z)$ belongs to $A_s^p(T_\Omega)$, $s=\sum\limits_{j=1}^{m}(\nu_j+\tau_j)+2(\dfrac{n}{r})(2m-1)$ and for every f function $f\in A_s^p$ there is an F function $F\in K_{\nu,\tau}^p$, so that $F(z,\dots,z)=f(z)$ for all $\dfrac{n}{r}\leqslant p_1,\ \dfrac{1}{p}+\dfrac{1}{p_1}=1$. Let in addition*

$$\Big(T_\beta f\Big)(z_1,\dots,z_m)=C_\beta\int_{T_\Omega}f(w)\prod_{j=1}^{m}\Delta^{-t}((z_j-\overline{w})/i)dV_\beta(w),$$

$mt=\beta+\dfrac{n}{r}$, $z_j\in T_\Omega$, $j=1,\dots,m$. Then the following assertions holds for all β, so that $\beta>\beta_0$ for some fixed large enough positive number β_0. The T_β Bergman type integral operator (expanded Bergman projection) maps $A_s^p(T_\Omega)$ to $K_{\nu,\tau}^p(T_\Omega^n)$, $\nu=(\nu_1,\dots,\nu_m)$, $\tau=(\tau_1,\dots,\tau_n)$, $\nu_j>\nu_0$, $\tau_j>\tau_0$, $j=1,...,m$.

Proof. Using arguments from the proof of the theorem 1 and applying lemma 5 and (7) we have the following chain of estimates

$$\int_{T_\Omega}|F(z,...,z)|^p\Delta^{r_1}(y)dydx=\sum_{j=1}^{\infty}\int_{B_j}|F(z,...,z)|^p\Delta^{r_1}(y)dydx\leqslant$$
$$\leqslant \widetilde{C}\sum_{j=1}^{\infty}\left(\sup_{z\in B_j}|F(z,...,z)|^p\right)\left(\int_{B_j}\Delta^{r_1}(y)dydx\right)\leqslant$$
$$\leqslant C\sum_{j_1=1}^{\infty}...\sum_{j_m=1}^{\infty}\sup_{\substack{z_1\in B_{j_1}\\ ...\\ z_m\in B_{j_m}}}|F(z_1,...,z_m)|^p\Delta^{r_2^1}(\operatorname{Im} z_1)...\Delta^{r_2^m}(\operatorname{Im} z_m)\leqslant$$
$$\leqslant C_1\sum_{j_1=1}^{\infty}...\sum_{j_m=1}^{\infty}\int_{B_{j_1}(z)}...\int_{B_{j_m}(z)}|F(\widetilde{z}_1,...,\widetilde{z}_m)|^p\Delta^{r_3^1}(\operatorname{Im}\widetilde{z}_1)...\Delta^{r_3^m}(\operatorname{Im}\widetilde{z}_m)d\widetilde{x}_1...d\widetilde{x}_m d\widetilde{y}_1...d\widetilde{y}_m\leqslant$$
$$\leqslant c\int_{T_\Omega}...\int_{T_\Omega}\left(\int_{B(z_1)}...\int_{B(z_m)}|F(w_1,...,w_m)|^p\prod_{j=1}^{m}\Delta^{\nu_j}(\operatorname{Im} w_j)\prod_{i=1}^{m}du_i dv_i\right)\times$$
$$\times\left[\prod_{j=1}^{m}\Delta^{\tau_j}(z_j)dx_j dy_j\right], w_j=u_j+iv_j, j=1,...,m$$

where $r_1=\sum\limits_{j=1}^{m}(\nu_j+\tau_j)+\dfrac{2n}{r}(2m-1),\ r_2^j=\nu_j+\tau_j+\dfrac{4n}{r},\ r_3^j=\nu_j+\tau_j+\dfrac{2n}{r},\ j=1,...,m.$

To get the reverse we again have to modify a little the proof of theorem 1. Namely we use the following additional estimate to get the result (see [19, 20])

$$\int_{B_j(w_j)}\frac{dV_\beta(w)}{\Delta^t\left(\frac{z_j-\overline{w}}{i}\right)}\leqslant\frac{C(\operatorname{Im} w_j)^{\beta+\frac{n}{r}}}{\Delta^t\left(\frac{z_j-\overline{w_j}}{i}\right)};\beta>\frac{n}{r}-1, t>0, z_j\in T_\Omega, w_j\in T_\Omega. \tag{8}$$

We omit easy technical details. We have for $\widetilde{\tau}=\left(\gamma_2 p_1 m-\beta-\dfrac{n}{r}\right)\left(\dfrac{p}{p_1 m}\right)$

$$|T_\beta f(z_1,...,z_m)|^p\leqslant C\int_{T_\Omega}\frac{|f(w)|^p dV_\beta(w)\prod\limits_{j=1}^{m}\Delta^{-\widetilde{\tau}}\operatorname{Im} z_j}{\prod\limits_{j=1}^{m}|\Delta^{\gamma_1 p}(\frac{z_j-\overline{w}}{i})|},\ \beta>\frac{n}{r}-1,$$

$$\gamma_1+\gamma_2=\left(\frac{n}{r}+\beta\right)\left(\frac{1}{m}\right),\ \gamma_2>\frac{\beta+\frac{2n}{r}-1}{p_1 m},$$

β is large enough, $\beta>\beta_0$.

Using (8) and then (5) m times we have after some calculations finally the estimate

$$\|T_\beta f\|_{K^p_{\nu,\tau}}\leqslant C\|f\|_{A^p_s}.$$

The proof of theorem 3 is complete. □

As we see proofs are based heavily on properties of r-lattices based on Bergman balls and related estimates(see [7, 8]). Complete analogues of all our assertions in disk, polyball can be found in [2–4, 14]. Moreover similar arguments we used in our proofs in polydisk and polyball can be seen in [2, 14].

Let

$$A^{\overrightarrow{p}}_{\overrightarrow{\nu}} = \{f \in H(T^m_\Omega) : \left(\int_{T_\Omega} ... \left(\int_{T_\Omega} |f(z_1, ..., z_m)|^{p_1} \left[\Delta^{\nu_1 - n/r}(y_1)\right] dx_1 dy_1\right)^{\frac{p_2}{p_1}} ...$$

$$... \left[\Delta^{\nu_m - n/r}(y_m)\right] dx_m dy_m\right)^{\frac{1}{p_m}} < \infty\}, \quad 1 \leqslant p_j < \infty, \nu_j > \frac{n}{r} - 1, j = 1, ..., m.$$

Some assertions of this note can be extended even to $A^{p,q}_\nu$ spaces in tubular domain T_Ω (see for definition the previous section) and to $A^{\overrightarrow{p}}_{\overrightarrow{\nu}}$ mixed norm spaces $p = (p_1, \ldots, p_m)$, $\nu = (\nu_1, \ldots, \nu_m)$ in products of tubular domains. For complete analogues of these mixed norm spaces in polyball we refer the reader to [6]. These results hovewer are not sharp. Note tubular domains are the most typical examples of general unbounded Siegel domains of second type.For the most typical examples of general bounded Siegel domains of second type the so-called bounded pseudoconvex domains with smooth boundary the complete analogues of all our results are also valid (see [21]).

Some results of this paper based on lemma 7 and 8, can be also easily extended to analitiec Besov B^p_ν spaces on product domains (see [9]).

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Точные теоремы о следах в аналитических пространствах в трубчатых областях над симметрическими конусами

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В статье предъявлены первые точные результаты о следах пространств Бергмана и типа Бергмана аналитических функций в трубчатых областях над симметрическими конусами.

Ключевые слова: аналитическая функция, трубчатая область, конус.