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Subharmonic Functions on Complex Hyperplanes of $\mathbb{C}^n$

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In this paper is considered a class of $m-wsh$ functions defined with relation $dd^c u \wedge (dd^c|z|^2)^{n-m} \geqslant 0$, and is studied some properties of polar sets for this class.

Keywords: $m-wsh$ function, mw-polar set, mw-convex domain, mw-regular domain.

Introduction

Subharmonic (sh) and plurisubharmonic (psh) functions play the main role in theory of functions of several real and complex variables. In the space $\mathbb{C}^n \approx \mathbb{R}^{2n}$ they defining by the conditions

$$dd^c u \wedge (dd^c\,|z|^2)^{n-1} \geqslant 0$$

or

$$dd^c u \geqslant 0,$$

respectively. Here, as usual $d = \partial + \overline{\partial}$, $d^c = \dfrac{\partial - \overline{\partial}}{4i}$.

In this paper we consider the class of m-weak subharmonic ($m-wsh$) functions, defined by relation

$$dd^c u \wedge (dd^c|z|^2)^{n-m} \geqslant 0. \qquad (1)$$

As we see below this class wider than the class of psh functions, but strongly contains in the class of sh functions. Moreover, in case, $m = 1$ the class of $1 - wsh$ functions coincide with class of sh functions and in case $m = n$ the class of $n-wsh$ functions coincide with class of psh functions.

In studying the class of m–wsh functions we essentially use the elementary theory of differential forms and currents, also methods of pluripotential theory. In general case, when u isn't twice differentiable, the relation (1) is interpretated in the sence of currents. Therefore in section 1 we shortly give fundamental conceptions from the theory of currents. In section 2 we give general definition of the $m-wsh$ functions and some their simple properties. Section 3 devoted to the mw-polar set and its characteristics.

1. Positive defined differential forms and currents

As usual, the space of differential forms of bidegree (p, p) in a domain $D \subset \mathbb{C}^n$ is denote by $\mathscr{F}^{(p,p)} = \mathscr{F}^{(p,p)}(D)$. The differential form in view

$$\omega = \left(\frac{i}{2}\right)^p (d\ell_1 \wedge d\bar{\ell}_1) \wedge ... \wedge (d\ell_p \wedge d\bar{\ell}_p)$$

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is called main positive form of bidegree (p,p), $0 \leqslant p \leqslant n$, where $\ell_j = a_{j_1} z_1 + ... + a_{j_n} z_n$ are linear functions in the space $\mathbb{C}^n$, $j = 1, 2, ..., p$. Linear combination of such form ω_q

$$\omega^{(p,p)} = \sum_{q=1}^{N} f_q(z)\omega_q, \quad f_q(z) \in C(D), \ \ f_q(z) \geqslant 0,$$

is called strongly positive differential form of bidegree (p,p) in the domain $D \subset \mathbb{C}^n$. Thus, positive differential form of bidegree $(0,0)$ or bidegree (n,n) give to us positive scalar function $\omega^{(0,0)} = f(z) \geqslant 0$ or

$$\omega^{(n,n)} = \left(\frac{i}{2}\right)^n f(z) dz_1 \wedge d\bar{z}_1 \wedge ... \wedge dz_n \wedge d\bar{z}_n = f(z)dV, \quad f(z) \geqslant 0,$$

where dV — Lebesgue's element of volume in the space $\mathbb{C}^n \simeq \mathbb{R}^{2n}$.

The differential forms $\omega^{(p,p)} \in \mathscr{F}^{(p,p)}$ of bidegree (p,p) is called weakly positive if $\omega^{(p,p)} \wedge \alpha$ is positive form of bidegree (n,n) for any strongly positive form $\alpha \in \mathscr{F}^{(n-p,n-p)}$. Strongly positive form is at the same time weakly positive, because exterior product of two strongly positive form are positive.

In the cases $p = 0, 1, n-1, n$ weakly and strongly positive are coincide. But, in cases $1 < p < n-1$ not every weakly positive differential form is strongly positive.

Definition 1. *Linear continuous functional $T(\omega)$ in the space of main differential form*

$$F^{(p,p)} = F^{(p,p)}(D) = \{\omega \in \mathscr{F}^{(p,p)}(D) \cap C^\infty(D) : \operatorname{supp}\omega \subset\subset D\}$$

is called current of bidegree $(n-p, n-p) = (q,q)$

The current T is called strongly (weakly) positive, if $T(\omega) \geqslant 0$ for any weakly (strongly) positive form $\omega \in \mathscr{F}^{(p,p)}$. It is clear, that for $q = 0, 1, n-1, n$ weakly positivity of currents also coincide with strongly positivity.

It is known, that positive currents are currents of measure type, i.e. differential forms, coefficients which are Borel's measures. More about the theory of currents see [1–4].

An impotent example of current of bidegree (p,p) in the pluripotential theory is current $dd^c u \wedge (dd^c|z|^2)^{p-1}$, $1 \leqslant p \leqslant n$, defined as

$$dd^c u \wedge \left(dd^c\,|z|^2\right)^{p-1}(\omega) = \int u\left(dd^c\,|z|^2\right)^{p-1} \wedge dd^c\omega\ ,\ \omega \in F^{(n-p,\,n-p)}(D), \tag{2}$$

where $u \in L^1_{loc}(D)$ are fixed functions. It is easy to proof that the current $dd^c u \wedge (dd^c|z|^2)^{p-1}$ is strongly positive if and only if, when it is weakly positive.

2. *m–wsh* functions

Definition 2. *A function $u(z) \in L^1_{loc}(D)$, given in a domain $D \subset \mathbb{C}^n$ is called $m-wsh$ function (subharmonic function on $(n-m+1)$–dimensional complex surfaces, $1 \leqslant m \leqslant n$) in D if:*

1) it is upper semicontinuous in D, i.e.

$$\varlimsup_{z \to z^0} u(z) = \lim_{\varepsilon \to 0} \sup_{B(z^0,\varepsilon)} u(z) \leqslant u(z^0);$$

2) the current $dd^c u \wedge (dd^c|z|^2|)^{n-m} \geqslant 0$ in D, i.e.

$$dd^c u \wedge \left(dd^c|z|^2\right)^{n-m}(\omega) = \int u\left(dd^c|z|^2\right)^{n-m} \wedge dd^c\omega \geqslant 0, \quad \forall\omega \in F^{(m-1,m-1)},\ \omega \geqslant 0.$$

The class of such functions is denoted by $m{-}wsh(D)$. For convenience, the function $u \equiv -\infty$ also included into the $m-wsh(D)$ class. A letter "w" (weak) in denotation of class is put in order to differ this class from the known class of $m-sh$ functions. $m-wsh$ function in the domain $D \subset \mathbb{C}^n$ at the same time is subharmonic in the $D \subset \mathbb{R}^{2n}$. Therefore, all properties of subharmonic functions is true for $m-wsh$ functions.

We provide a following properties of $m-wsh$ function, which we will use further.

1) Linear combination of m–wsh functions with nonnegative coefficients are m–wsh functions, i.e.

$$u_j(z) \in m - wsh(D), \quad a_j \in R_+ \;\; (j = 1, 2, ..., N) \quad \Rightarrow$$

$$a_1u_1(z) + a_2u_2(z) + ... + a_Nu_N(z) \in m{-}wsh(D).$$

2) A limit of monotonically decreasing sequences of $m-wsh$ functions is $m-wsh$ function, i.e.

$$u_j(z) \in m - wsh(D), \quad u_j(z) \geq u_{j+1}(z), \quad (j = 1, 2, ...) \quad \Rightarrow$$

$$\lim_{j\to\infty} u_j(z) \in m{-}wsh(D).$$

3) Uniformly convergence of sequence of $m-wsh$ functions is converge to $m-wsh$ function, i.e. if $u_j(z) \in m{-}wsh(D)$, $(j = 1, 2, ...)$, $u_j(z) \rightrightarrows u(z)$, then $u(z) \in m{-}wsh(D)$.

4) *(maximum principle).* Let a function $u(z) \in m-wsh(D)$ and in some point $z^0 \in D$ it reaches its maximum, i.e.

$$u(z^0) = \sup_{z\in D} u(z). \tag{3}$$

Then $u(z) \equiv \text{const}$.

5) If $u(z) \in m$–$wsh(D)$, then a convolution $u_j(z) = u{*}K_{1/j}(z{-}w)$ also belongs to $m{-}wsh(D)$, and $u_j(z) \downarrow u(z)$ at $j \to \infty$.

Here $K_{1/j}(x) = j^nK(jx)$ and K is standard infinity differentiable kernel, with carrier $\text{supp}K \subset B(0, 1)$ and

$$\int_{R^n} K(x)dx = \int_{B(0,1)} K(x)dx = 1.$$

The proof of these properties implies from analogous properties of subharmonic functions on the plane and we down them (in details see [5]).

A following theorem gives us geometric character of $m-wsh$ functions.

Theorem 1. *Upper semi continuous function u, given in the domain $D \subset \mathbb{C}^n$, is $m-wsh$ if and only if for any $(n - m + 1)$–dimensional complex surface $\Pi \subset \mathbb{C}^n$ restriction*

$$u|_{\Pi} \in sh\,(\Pi \cap D)\,. \tag{4}$$

Proof. *Necessity.* Let $u \in m - wsh(D)$. According to property 5 we approximate u, with infinity differentiable functions $u_j \downarrow u$, $u_j \in m - wsh(D) \cap C^\infty(D)$. We fix a complex plane $\Pi \subset \mathbb{C}^n$, $\dim_C \Pi = n - m + 1$, and we take an orthonormal basis $\xi_1, ..., \xi_{n-m+1}$ on Π. Then $(dd^c|z|^2)^{n-m}|_\Pi = (dd^c|\xi|^2)^{n-m}$ and consequently, $dd^cu_j \wedge (dd^c|z|^2)^{n-m}\,|_\Pi = dd^cu_j\,|_\Pi \wedge (dd^c|\xi|^2)^{n-m}$. Since, $dd^cu_j \wedge (dd^c|z|^2)^{n-m}$ is positive differential form of bidegree $(n - m + 1, n - m + 1)$, then the restriction $dd^cu_j \wedge (dd^c|z|^2)^{n-m}\,|_\Pi \geqslant 0$. Hence $dd^cu_j\,|_\Pi \wedge (dd^c|\xi|^2)^{n-m} \geqslant 0$ and it means, that $u_j\,|_\Pi \in sh(\Pi \cap D)$. Since, $u_j\,|_\Pi \downarrow u\,|_\Pi$ at $j \to \infty$, then $u\,|_\Pi \in sh(D)$.

Sufficiency. First we formulate a number of properties of upper semi continuous function $u(z)$, satisfying the condition (4), by them we will proof of sufficiency of theorem.

1) Finite sum $\alpha_1u_1 + ... + \alpha_ku_k$ with positive coefficients $\alpha_1, ..., \alpha_k \geqslant 0$ will satisfy the condition (4), if and only if $u_1, ..., u_k$ satisfy the condition (4).

2) Decreasing sequence or uniformly convergence sequence of functions $\{u_j\}$, satisfying the condition (4) converges to function of type (4).

3) The function u, satisfying the condition (4) either $u \equiv -\infty$, or locally summable function, i.e. $u \in L^1_{loc}(D)$.

Indeed, since u is upper semicontinuous, then it locally bounded from above. Therefore, without lost of generality we may assume, that $u < 0$ in D. Let in some point $z^0 = 0$ the function $u(0) \neq -\infty$. Then for any fixed surface $\Pi \ni 0$, $\dim \Pi = n - m + 1$, the restriction $u\,|_\Pi$ is subharmonic in $D \cap \Pi$. Consequently,

$$u\,(0) \leqslant \frac{1}{V_{n-m+1} r^{n-m+1}} \int_{B(0,r)\cap\Pi} u\,|_\Pi \, dV|_\Pi, \tag{5}$$

where $B(0, r) = \{\|z\| < r\}$ is a ball, $dV|_\Pi$ is an element of volume on Π and V_{n-m+1} is a volume of unit ball in $\Pi \simeq \mathbb{R}^{n-m+1}$. Hence, for any surface $\Pi \ni 0$, $\dim \Pi = n-m+1$, the restriction $u\,|_\Pi$ has uniformly bounded integrals on $\Pi \cap B(0, r)$. By the Fubini theorem and according to (5) it follows that, $-\infty < \int_{B(0,r)} u(z)dz < 0$. It means, that u locally integrable in a neighbourhood of origin and it follows that the function u integrable on any Ball $B(z^0, r)$, $z^0 \in D$, $r > 0$.

Remark 1. Here we apply the Fubini theorem on collection of complex surfaces passing through origin. As it is known they generate Grassman's manifold $M_{n,n-m+1}$. But to prove locally integrability of u we can apply the theorem of Fubini for all complex surfaces Π passing through some fixed surface $L \ni 0$, $\dim L = n - m$. The set of such Π will generate a projective space P^{m-1}, and to proof $u \in L^1_{loc}(D)$ we can use a following convenient formula of Fubini

$$\int_{B(0,r)} u(z)\, dv = \int_{\Pi\in\mathrm{P}^{m-1}} \omega^{m-1} \int_{B(0,r)\cap\Pi} u|_\Pi(z)\, dV|_\Pi, \tag{6}$$

where ω is standard form of Fubini-Shtudi of projective space.

4) If u satisfy the condition (4), then the convolution $u_j(z) = u * K_{1/j}(z - w)$ also satisfy this condition and $u_j(z) \downarrow u(z)$ at $j \to \infty$.

It follows from obviously relation

$$u * K_{1/j}(z - w) = j^n \int_{R^n} u(w)K(j(z - w))dw = \int_{R^n} u\left(z + \frac{w}{j}\right) K(w)dw. \tag{7}$$

Here, the first integral represents infinity differentiable function, second integral satisfies the condution (4). Convergence of $u_j(z) \downarrow u(z)$ follows from (6).

Now we can complete the proof of theorem1. According to property 4) we construct approximation $u_j(z) \downarrow u(z)$. Since, $u_j \in C^\infty$ and $u_j|_\Pi$ are subharmonic on any complex surface Π, $\dim_C \Pi = n - m + 1$, then the restriction $dd^c u_j \wedge (dd^c|z|^2)^{n-m}\,|_\Pi \geqslant 0$. It means, that the differential form $dd^c u_j \wedge (dd^c|z|^2)^{n-m} \geqslant 0$. From convergence of $u_j(z) \downarrow u(z)$ follows $dd^c u \wedge (dd^c|z|^2)^{n-m} \geqslant 0$ in the sence of currents, and consequently, $u \in m{-}wsh(D)$. The proof of theorem1 is complete. □

3. *mw*-polar sets

The polar and pluripolar sets are key notions of the potential theory (see [3, 6]). Therefore, it is important the study of the *mw*-polar sets for the class of *msh*-functions.

Definition 3. *By analogue polar sets, a set $E \subset D \subset \mathbb{C}^n$ is called mw-polar in D, if there exist a function $u(z) \in m{-}wsh(D)$, $u(z) \not\equiv -\infty$, such that $u\,|_E = -\infty$.*

From inclusion $m-wsh(D) \subset sh(D)$ it is follows, that each mw-polar set is polar. In partiqular, the Hausdorf measure $\mathrm{H}_{2n-2+\varepsilon}(E) = 0\ \forall\varepsilon > 0$, and consequently, Lebesgue measure of mw-polar set E also is zero.

From embedding $psh(D) \subset m-wsh(D)$ follows, that every pluripolar set is mw-polar. We provide a nontrivial example of mw-polar set in the space $\mathbb{C}^3$.

Example 1. We consider a function

$$u = \ln[(z_1+\bar z_1)^2 + (z_2+\bar z_2)^2 + (z_3+\bar z_3)^2] = \ln|z+\bar z|^2 = \ln\left(x_1^2+x_2^2+x_3^2\right) + \ln 4,$$

where $z_j = x_j + iy_j, j = 1,2,3$.

It is clear u is not $3-wsh$ in D, i.e. it is not psh in D. It is not difficult to prove that it is subharmonic, i.e. $\Delta u \geqslant 0$. We show that it is $2-wsh$ function in $\mathbb{C}^3$. Thereby we have, that real 3-dimentional surface $\mathbb{R}^3(x) = \left\{z\in\mathbb{C}^3 : Imz = 0\right\}$ is $2w$-polar in $\mathbb{C}^3$. Taking direct calculation.

$$\omega = (dd^c u)\wedge dd^c|z|^2 = \frac{i}{2}\left[\frac{\partial^2 u}{\partial z_1\partial\bar z_1}\,dz_1\wedge d\bar z_1 + \frac{\partial^2 u}{\partial z_1\partial\bar z_2}dz_1\wedge d\bar z_2 + \right.$$

$$+\frac{\partial^2 u}{\partial z_1\partial\bar z_3}dz_1\wedge d\bar z_3 + \frac{\partial^2 u}{\partial z_2\partial\bar z_1}dz_2\wedge d\bar z_1 + \frac{\partial^2 u}{\partial z_2\partial\bar z_2}dz_2\wedge d\bar z_2 + \frac{\partial^2 u}{\partial z_2\partial\bar z_3}dz_2\wedge d\bar z_3 +$$

$$\left. +\frac{\partial^2 u}{\partial z_3\partial\bar z_1}dz_3\wedge d\bar z_1 + \frac{\partial^2 u}{\partial z_3\partial\bar z_2}dz_3\wedge d\bar z_2 + \frac{\partial^2 u}{\partial z_3\partial\bar z_3}dz_3\wedge d\bar z_3\right]\wedge$$

$$\wedge\frac{i}{2}(dz_1\wedge d\bar z_1 + dz_2\wedge d\bar z_2 + dz_3\wedge d\bar z_3) = -\frac{1}{4}\left[\left(\frac{\partial^2 u}{\partial z_1\partial\bar z_1} + \frac{\partial^2 u}{\partial z_2\partial\bar z_2}\right)dz_1\wedge d\bar z_1\wedge dz_2\wedge d\bar z_2 + \right.$$

$$+\left(\frac{\partial^2 u}{\partial z_1\partial\bar z_1} + \frac{\partial^2 u}{\partial z_3\partial\bar z_3}\right)dz_1\wedge d\bar z_1\wedge dz_3\wedge d\bar z_3 + \left(\frac{\partial^2 u}{\partial z_2\partial\bar z_2} + \frac{\partial^2 u}{\partial z_3\partial\bar z_3}\right)dz_2\wedge d\bar z_2\wedge dz_3\wedge d\bar z_3 +$$

$$+\frac{\partial^2 u}{\partial z_2\partial\bar z_3}dz_1\wedge d\bar z_1\wedge dz_2\wedge d\bar z_3 + \frac{\partial^2 u}{\partial z_3\partial\bar z_2}dz_1\wedge d\bar z_1\wedge dz_3\wedge d\bar z_2 + \frac{\partial^2 u}{\partial z_1\partial\bar z_3}dz_1\wedge d\bar z_3\wedge dz_2\wedge d\bar z_2 +$$

$$\left.+\frac{\partial^2 u}{\partial z_3\partial\bar z_1}dz_3\wedge d\bar z_1\wedge dz_2\wedge d\bar z_2 + \frac{\partial^2 u}{\partial z_1\partial\bar z_2}dz_1\wedge d\bar z_2\wedge dz_3\wedge d\bar z_3 + \frac{\partial^2 u}{\partial z_2\partial\bar z_1}dz_2\wedge d\bar z_1\wedge dz_3\wedge d\bar z_3\right].$$

Thus, for any form $\nu = \frac{i}{2}d\ell\wedge d\bar\ell$ of bidegree $(1,1)$ where $d\ell = a_1dz_1 + a_2dz_2 + a_3dz_3$ from $\nu = \frac{i}{2}d\ell\wedge d\bar\ell = \frac{i}{2}\Big(|a_1|^2 dz_1\wedge d\bar z_1 + a_1\bar a_2 dz_1\wedge d\bar z_2 + a_1\bar a_3 dz_1\wedge d\bar z_3 + a_2\bar a_1 dz_2\wedge d\bar z_1 + |a_2|^2 dz_2\wedge d\bar z_2 + a_2\bar a_3 dz_2\wedge d\bar z_3 + a_3\bar a_1 dz_3\wedge d\bar z_1 + a_3\bar a_2 dz_3\wedge d\bar z_2 + |a_3|^2 dz_3\wedge d\bar z_3\Big)$, we get

$$\nu\wedge\omega = -\frac{i}{8}\left[|a_1|^2\left(\frac{\partial^2 u}{\partial z_2\partial\bar z_2} + \frac{\partial^2 u}{\partial z_3\partial\bar z_3}\right)dz_1\wedge d\bar z_1\wedge dz_2\wedge d\bar z_2\wedge dz_3\wedge d\bar z_3 + \right.$$

$$+a_1\bar a_2\frac{\partial^2 u}{\partial z_2\partial\bar z_1}dz_1\wedge d\bar z_2\wedge dz_2\wedge d\bar z_1\wedge dz_3\wedge d\bar z_3 + a_1\bar a_3\frac{\partial^2 u}{\partial z_3\partial\bar z_1}dz_1\wedge d\bar z_3\wedge dz_3\wedge d\bar z_1\wedge dz_2\wedge d\bar z_2 +$$

$$+a_2\bar a_1\frac{\partial^2 u}{\partial z_1\partial\bar z_2}dz_2\wedge d\bar z_1\wedge dz_1\wedge d\bar z_2\wedge dz_3\wedge d\bar z_3 + |a_2|^2\left[\frac{\partial^2 u}{\partial z_1\partial\bar z_1} + \frac{\partial^2 u}{\partial z_3\partial\bar z_3}\right]dz_1\wedge d\bar z_1\wedge dz_2\wedge d\bar z_2\wedge dz_3\wedge d\bar z_3 +$$

$$+a_2\bar a_3\frac{\partial^2 u}{\partial z_3\partial\bar z_2}dz_2\wedge d\bar z_3\wedge dz_1\wedge d\bar z_1\wedge dz_3\wedge d\bar z_2 + a_3\bar a_1\frac{\partial^2 u}{\partial z_1\partial\bar z_3}dz_3\wedge d\bar z_1\wedge dz_1\wedge d\bar z_2\wedge dz_2\wedge d\bar z_3 +$$

$$+a_3\bar a_2\frac{\partial^2 u}{\partial z_2\partial\bar z_3}dz_3\wedge d\bar z_2\wedge dz_1\wedge d\bar z_1\wedge dz_2\wedge d\bar z_3 + |a_3|^2\left[\frac{\partial^2 u}{\partial z_1\partial\bar z_1} + \frac{\partial^2 u}{\partial z_2\partial\bar z_2}\right]dz_1\wedge d\bar z_1\wedge dz_2\wedge d\bar z_2\wedge dz_3\wedge d\bar z_3 =$$

$$= \left[|a_1|^2\left(\frac{\partial^2 u}{\partial z_2\partial\bar z_2}+\frac{\partial^2 u}{\partial z_3\partial\bar z_3}\right)+|a_2|^2\left(\frac{\partial^2 u}{\partial z_1\partial\bar z_1}+\frac{\partial^2 u}{\partial z_3\partial\bar z_3}\right)+|a_3|^2\left(\frac{\partial^2 u}{\partial z_1\partial\bar z_1}+\frac{\partial^2 u}{\partial z_2\partial\bar z_2}\right)-\right.$$
$$\left.-a_1\bar a_2\frac{\partial^2 u}{\partial z_2\partial\bar z_1}-a_1\bar a_3\frac{\partial^2 u}{\partial z_3\partial\bar z_1}-a_2\bar a_1\frac{\partial^2 u}{\partial z_1\partial\bar z_2}-a_2\bar a_3\frac{\partial^2 u}{\partial z_3\partial\bar z_2}-a_3\bar a_1\frac{\partial^2 u}{\partial z_1\partial\bar z_3}-a_3\bar a_2\frac{\partial^2 u}{\partial z_2\partial\bar z_3}\right]\times$$
$$\times\frac{i}{2}dz_1\wedge d\bar z_1\wedge\frac{i}{2}dz_2\wedge d\bar z_2\wedge\frac{i}{2}dz_3\wedge d\bar z_3=\alpha(z)\frac{i}{2}dz_1\wedge d\bar z_1\wedge\frac{i}{2}dz_2\wedge d\bar z_2\wedge\frac{i}{2}dz_3\wedge d\bar z_3,$$

where

$$\alpha(z)=|a_1|^2\left(\frac{2\,|z+\bar z|^2-4(z_2+\bar z_2)^2}{|z+\bar z|^4}+\frac{2\,|z+\bar z|^2-4(z_3+\bar z_3)^2}{|z+\bar z|^4}\right)+$$
$$+|a_2|^2\left(\frac{2\,|z+\bar z|^2-4(z_1+\bar z_1)^2}{|z+\bar z|^4}+\frac{2\,|z+\bar z|^2-4(z_3+\bar z_3)^2}{|z+\bar z|^4}\right)+$$
$$+|a_3|^2\left(\frac{2\,|z+\bar z|^2-4(z_1+\bar z_1)^2}{|z+\bar z|^4}+\frac{2\,|z+\bar z|^2-4(z_2+\bar z_2)^2}{|z+\bar z|^4}\right)+$$
$$+a_1\bar a_2\frac{4(z_1+\bar z_1)(z_2+\bar z_2)}{|z+\bar z|^4}+a_1\bar a_3\frac{4(z_1+\bar z_1)(z_3+\bar z_3)}{|z+\bar z|^4}+$$
$$+a_2\bar a_1\frac{4(z_1+\bar z_1)(z_2+\bar z_2)}{|z+\bar z|^4}+a_2\bar a_3\frac{4(z_2+\bar z_2)(z_3+\bar z_3)}{|z+\bar z|^4}+$$
$$+a_3\bar a_1\frac{4(z_1+\bar z_1)(z_3+\bar z_3)}{|z+\bar z|^4}+a_3\bar a_2\frac{4(z_2+\bar z_2)(z_3+\bar z_3)}{|z+\bar z|^4}=$$
$$=\frac{4}{|z+\bar z|^4}\Big(|a_1|^2(z_1+\bar z_1)^2+|a_2|^2(z_2+\bar z_2)^2+|a_3|^2(z_3+\bar z_3)^2+a_1\bar a_2(z_1+\bar z_1)(z_2+\bar z_2)+$$
$$+a_1\bar a_3(z_1+\bar z_1)(z_3+\bar z_3)+a_2\bar a_1(z_2+\bar z_2)(z_1+\bar z_1)+a_2\bar a_3(z_2+\bar z_2)(z_3+\bar z_3)+$$
$$+a_3\bar a_1(z_3+\bar z_3)(z_1+\bar z_1)+a_3\bar a_2(z_2+\bar z_2)(z_3+\bar z_3)\Big)=$$
$$=\frac{4}{|z+\bar z|^4}\left|a_1(z_1+\bar z_1)+a_2(z_2+\bar z_2)+a_3(z_3+\bar z_3)\right|^2\geqslant 0.$$

Since, ℓ-arbitrary linear function, then $dd^cu\wedge dd^c|z|^2\geqslant 0$, in $\mathbb{C}^3\backslash\mathbb{R}^3(x)$ i.e. u is $2-wsh$ function beyond of points $\mathbb{R}^3(x)$. In points $\mathbb{R}^3(x)$ function $u\big|_{\mathbb{R}^3(x)}=-\infty$. Consequently, it will be automatically $2-wsh$ in these sense.

Definition 4. *A domain $D\subset\mathbb{C}^n$ is called mw-convex, if there exist $\rho(z)\in m-wsh(D)$ such that $\lim\limits_{z\to\partial D}\rho(z)=+\infty$, and it called mw-regular, if there exist $\rho(z)\in m-wsh(D):\rho(z)<0$ such that $\lim\limits_{z\to\partial D}\rho(z)=0$.*

Next two theorems are analogue of corresponding theorems of classical and complex theory of potential (see for example [6,7]).

Theorem 2. *Countable union of mw-polar sets is mw-polar, i.e. if $E_j\subset D$ are mw-polar, then $E=\bigcup\limits_{j=1}^{\infty}E_j$ is also mw-polar.*

Theorem 3. *Let $D\subset\mathbb{C}^n$ be mw-convex domain and subset $E\subset D$ such that for any compact subdomain $G\subset\subset D$ the set $E\cap G$ mw-polar in G. Then E is mw-polar in D. Moreover, if D–mw is regular, then there exist a function $u(z)\in m-wsh(D)$, $u|_D<0$, $u\not\equiv-\infty$, but $u|_E\equiv-\infty$.*

Proofs of these theorem close to eachother. Therefore we provide only proof of the Theorem 3.

Since D is $mw-$ convex domain, then a function $\rho(z) = -\ln\rho(z,\partial D)$ is $m-wsh(D)$ and $\lim\limits_{z\to\partial D}\rho(z) = +\infty$. Hence, $D_r = \{z \in \partial D : \rho(z) < r\} \subset\subset D$ for any $r > 0$. We fix some point $a \in D$ and denote by G_j connected component of the set D_{r_j}, enclosed a point a. Then there exist a number $r_j > \rho(a)$ such that

$$G_j \subset\subset G_{j+1}, \quad \bigcup_{j=1}^{\infty} G_j = D. \tag{8}$$

Since $E \cap G_{j+1}$ is mw-polar, then there exist a functions $v_j(z) \in m-wsh(G_{j+2})$ such that $v_j \not\equiv -\infty$, but $v_j\big|_{E\cap G_{j+2}} \equiv -\infty$. As the set $\{v_j = -\infty\}$ has a Lebesgue measure zero, then the set $\bigcup\limits_{j=1}^{\infty}\{v_j = -\infty\}$ also has a Lebesgue measure zero. Consequently, there is a point $z^0 \in G_1$ such that $v_j(z^0) \neq -\infty$ for all $j \in N$.

Putting $C_j = \max\limits_{z\in\bar{G}_{j+1}} v_j(z)$, $\widehat{v}_j(z) = -\dfrac{1}{2^j}\cdot\dfrac{v_j(z)-C_j}{v_j(z^0)-C_j}$ and $u_j(z) = a_j(\rho(z) - r_{j+1})$, where $a_j > 0$ so big, that $u\big|_{G_j} \leqslant -1$. Then $\widehat{v}_j(z)\big|_{G_{j-1}} < 0$ and $u_j\big|_{\partial G_{j+1}} \equiv 0$. Therefore, it is not difficult to proof, that

$$w_j(z) = \begin{cases} \max\{\widehat{v}_j(z), u_j(z)\}, & \text{for } z \in G_{j+1}, \\ u_j(z), & \text{for } z \notin G_{j+1} \end{cases} \tag{9}$$

is mw-subharmonic in D $(j = 1, 2, ...)$.

Then the sum $w(z) = \sum\limits_{j=1}^{\infty} w_j(z) \in m-wsh(D)$, and $w\left(z^0\right) = -1, w\,|_E \equiv -\infty$. It follows that E is mw-polar in D.

In the case, when $D = \{\rho(z) < 0\}$ is mw-regular, i.e. $\rho(z) \in m-wsh(D) : \rho(z) < 0$ and $\lim\limits_{z\to\partial D}\rho(z) = 0$, as a set $D_r = \{z \in \partial D :\ \rho(z) < -r\} \subset\subset D, r > 0$, and as a function u_j we take $u_j(z) = a_j[\rho(z) + r_{j+1}]$. Here the sequence $r_j \downarrow 0$ such, that the connected component G_j of D_{r_j} satisfy the condition (8) and the a_j a such, that $u\big|_{G_j} \leqslant -1$. Further, we construe w_j as in (9) and we put $w(z) = \sum\limits_{j=1}^{\infty} w_j(z)$. Then w will be at first negative $m-wsh$ function in D and secondly $w\,|_E \equiv -\infty$. □

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Субгармонические функции на комплексных гиперплоскостях $\mathbb{C}^n$

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В данной статье рассмотрен класс $m-wsh$ функций, определяемых соотношением $dd^c u \wedge$ $\wedge (dd^c|z|^2)^{n-m} \geqslant 0$, и изучены некоторые свойства полярных множеств из этого класса.

Ключевые слова: $m-wsh$ функции, mw-полярное множество, mw-выпуклая область, mw-регулярная область.